TRANSFORMADA DE LAPLACE
Ampliación de Matemáticas
Ingeniero de Telecomunicación

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1. Utilizar la identidad $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$ para obtener las Transformadas de Laplace de $\cos^2 kt$ y $\sin^2 kt$

2. Determinar $\mathcal{L}[f(t)](z)$ donde

   (a) $f(t) = \begin{cases} 4 & \text{si} \quad 0 \le t < 1, \\ 3 & \text{si} \quad t \ge 1. \end{cases}$

   (b) $f(t) = \begin{cases} 1 & \text{si} \quad 0 \le t < 2, \\ t & \text{si} \quad t \ge 2. \end{cases}$

   (c) $f(t) = \begin{cases} 0 & \text{si} \quad 0 \le t < 1, \\ t & \text{si} \quad 1 \le t < 2, \\ 0 & \text{si} \quad t \ge 2. \end{cases}$

   (d) $f(t) = \begin{cases} \sin 2t & \text{si} \quad 0 \le t < \pi, \\ 0 & \text{si} \quad t \ge \pi. \end{cases}$

3. La función de onda triangular, que denotaremos por $T(t,c)$, viene dada por

$$T(t,c) = \begin{cases} t & \text{si} \quad 0 \le t < c, \\ 2c - t & \text{si} \quad c \le t < 2c \end{cases} \qquad \text{y} \qquad T(t+2c,c) = T(t,c)$$

   Dibujar esta función y determinar su Transformada de Laplace.

4. Dibujar la gráfica de la función $g(t) = e^t$ siendo $0 \le t < c$ tal que $g(t+c) = g(t)$. Determinar su Transformada de Laplace.

5. Dibujar la gráfica de la función $h(t) = 1 - t$ siendo $0 \le t < 1$ tal que $h(t+1) = h(t)$. Determinar su Transformada de Laplace.

6. Determinar la Transformada Inversa de Laplace de las siguientes funciones

   (a) $F_1(z) = \dfrac{1}{z^2 + 2z + 10}$.

   (b) $F_2(z) = \dfrac{1}{z^2 - 4z + 8}$.

(c) $F_3(z) = \dfrac{1}{z^2 + 4z + 13}$.

(d) $F_4(z) = \dfrac{z}{z^2 + 6z + 13}$.

(e) $F_5(z) = \dfrac{1}{z^2 + 4z + 4}$.

(f) $F_6(z) = \dfrac{z}{z^2 + 4z + 4}$.

(g) $F_7(z) = \dfrac{2z - 3}{z^2 - 4z + 8}$.

(h) $F_8(z) = \dfrac{3z + 1}{z^2 + 6z + 13}$.

(i) $F_9(z) = \dfrac{2z + 3}{(z + 4)^3}$.

(j) $F_{10}(z) = \dfrac{z^2}{(z - 1)^4}$.

7. Determinar la Transformada Inversa de Laplace de las siguientes funciones donde $a^2 \neq b^2$ y $ab \neq 0$.

(a) $G_1(z) = \dfrac{1}{z^2 + az}$.

(b) $G_2(z) = \dfrac{2z^2 - 1}{z(z + 1)^2}$.

(c) $G_3(z) = \dfrac{2z^2 + 5z - 4}{z^3 + z^2 - 2z}$.

(d) $G_4(z) = \dfrac{4z + 4}{z^2(z - 2)}$.

(e) $G_5(z) = \dfrac{5z - 2}{z^2(z + 2)(z - 1)}$.

(f) $G_6(z) = \dfrac{1}{z^3(z^2 + 1)}$.

(g) $G_7(z) = \dfrac{2s - 3}{z^2 - 4s + 8}$.

(h) $G_8(z) = \dfrac{1}{(z^2 + a^2)(z^2 + b^2)}$.

(i) $G_9(z) = \dfrac{z}{(z^2 + a^2)(z^2 + b^2)}$.

(j) $G_{10}(z) = \dfrac{z^2}{(z^2 + a^2)(z^2 + b^2)}$.

8. Resolver cada uno de los siguientes problemas de valor inicial por medio del método de la Transformada de Laplace. Verificar la solución.

(a) $\begin{cases} y' = e^t, \\ y(0) = 2. \end{cases}$

(b) $\begin{cases} y' - y = e^{-t}, \\ y(0) = 1. \end{cases}$

(c) $\begin{cases} y'' + a^2 y = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$

(d) $\begin{cases} y'' - 3y' + 2y = e^{3t}, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(e) $\begin{cases} y'' + y = e^{-t}, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(f) $\begin{cases} y'' + y' - 2y = -4, \\ y(0) = 2, \\ y'(0) = 3. \end{cases}$

(g) $\begin{cases} y'' - 4y' + 4y = 4e^{2t}, \\ y(0) = -1, \\ y'(0) = -4. \end{cases}$

(h) $\begin{cases} y'' + 9y = 40e^t, \\ y(0) = 5, \\ y'(0) = -2. \end{cases}$

(i) $\begin{cases} y'' + 3y' + 2y = 4t^2, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(j) $\begin{cases} \frac{1}{4}y'' - y' + y = \cos 2t, \\ y(0) = 2, \\ y'(0) = 5. \end{cases}$

9. Representar la gráfica de las siguientes funciones para $t \geq 0$

(a) $f_1(t) = H(t-a)$.

(b) $f_2(t) = (t-3)\, H(t-3)$.

(c) $f_3(t) = t^2 - t^2 H(t-2)$.

(d) $f_4(t) = \sin(t-\pi)\, H(t-\pi)$.

(e) $f_5(t) = (t-3)^2\, H(t-3)$.

(f) $f_6(t) = t^2 - (t-1)^2\, H(t-1)$.

10. Determinar la Transformada de Laplace de las siguientes funciones utilizando la función de Heaviside

(a) $f_1(t) = \begin{cases} 3 & \text{si} \quad 0 \leq t < 1, \\ t & \text{si} \quad t \geq 1. \end{cases}$

(b) $f_2(t) = \begin{cases} 4 & \text{si} \quad 0 \leq t < 2, \\ 2t - 1 & \text{si} \quad t \geq 2. \end{cases}$

(c) $f_3(t) = \begin{cases} t^2 & \text{si} \quad 0 \leq t < 2, \\ 3 & \text{si} \quad t \geq 2. \end{cases}$

(d) $f_4(t) = \begin{cases} e^{-t} & \text{si} \quad 0 \leq t < 2, \\ 0 & \text{si} \quad t \geq 2. \end{cases}$

(e) $f_5(t) = \begin{cases} \sin 3t & \text{si} \quad 0 \leq t < \frac{1}{2}\pi, \\ 0 & \text{si} \quad t \geq \frac{1}{2}\pi. \end{cases}$

(f) $f_6(t) = \begin{cases} \sin 3t & \text{si} \quad 0 \leq t < \pi, \\ 0 & \text{si} \quad t \geq \pi. \end{cases}$

(g) $f_7(t) = \begin{cases} t^2 & \text{si} \quad 0 \leq t < 1, \\ 3 & \text{si} \quad 1 \leq t < 2, \\ 0 & \text{si} \quad t \geq 2. \end{cases}$

(h) $f_8(t) = \begin{cases} t^2 & \text{si} \quad 0 \leq t < 2, \\ t - 1 & \text{si} \quad 2 \leq t < 3, \\ 7 & \text{si} \quad t \geq 3. \end{cases}$

11. Determinar

(a) $\mathcal{L}^{-1}\left[\frac{5e^{-3z}}{z} - \frac{e^{-z}}{z}\right](t)$.

(b) $\mathcal{L}^{-1}\left[\frac{e^{-4z}}{(z+2)^3}\right](t)$.

(c) $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{(z+1)^3}\right](t)$.

(d) $\mathcal{L}^{-1}\left[\frac{\left(1-e^{-2s}\right)\left(1-3e^{-2s}\right)}{z^2}\right](t)$.

12. Resolver los siguientes problemas de valor inicial usando la Transformada de Laplace. Verificar la solución.

(a) $\begin{cases} y'' + y = f_1(t), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$ donde $f_1(t) = \begin{cases} 4 & \text{si } 0 \le t < 2, \\ t+2 & \text{si } t \ge 2. \end{cases}$

(b) $\begin{cases} y'' + y = f_2(t), \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$ donde $f_2(t) = \begin{cases} 3 & \text{si } 0 \le t < 4, \\ 2t-5 & \text{si } t \ge 4. \end{cases}$

(c) $\begin{cases} y'' + y = f_3(t), \\ x(0) = 0, \\ x'(0) = 1. \end{cases}$ donde $f_3(t) = \begin{cases} 1 & \text{si } 0 \le t < \frac{\pi}{2}, \\ 0 & \text{si } t \ge \frac{\pi}{2}. \end{cases}$

(d) $\begin{cases} y'' + 4y = f_4(t), \\ x(0) = 0, \\ x'(0) = 0. \end{cases}$ donde $f_4(t) = \sin t - H(t-2\pi)\sin(t-2\pi)$.

13. Determinar $y\left(\frac{1}{2}\pi\right)$ y $y\left(2+\frac{1}{2}\pi\right)$ para la función $y(t)$ que satisface el problema de valor inicial

$$\begin{cases} y'' + y = (t-2)\,H(t-2), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

14. Determinar $y(1)$ y $y(4)$ para la función $y(t)$ que satisface el problema de valor inicial

$$\begin{cases} y'' + +2y' + y = 2 + (t-3)\,H(t-3), \\ y(0) = 2, \\ y'(0) = 1. \end{cases}$$

15. Determinar las siguientes Transformadas de Laplace mediante el Teorema de convolución

(a) $F_1(z) = \dfrac{1}{z(z^2+k^2)}$.

(b) $F_2(z) = \dfrac{4}{z^2(z-2)}$.

(c) $F_3(z) = \dfrac{1}{z(z+2)}$.

(d) $F_4(z) = \dfrac{1}{(z^2+1)^2}$.

16. Resolver los siguientes problemas de valor inicial utilizando el Teorema de convolución.

(a) $\begin{cases} y'' + 2y' + y = f_1(t), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(b) $\begin{cases} y'' - k^2 y = f_2(t), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(c) $\begin{cases} y'' + 4y' + 13y = f_3(t), \\ x(0) = 0, \\ x'(0) = 0. \end{cases}$

(d) $\begin{cases} y'' + 6y' + 9y = f_4(t), \\ x(0) = A, \\ x'(0) = B. \end{cases}$

17. Resolver las siguientes ecuaciones

(a) $f_1(t) = 1 + 2\int_0^t f_1(t-s)\, e^{-2s} ds.$

(b) $f_2(t) = 1 + \int_0^t f_2(s) \sin(t-s)\, ds.$

(c) $f_3(t) = t + \int_0^t f_3(t-s)\, e^{-s} ds.$

(d) $f_4(t) = 4t^2 - \int_0^t f_4(t-s)\, e^{-s} ds.$

(e) $f_5(t) = t^3 + \int_0^t f_5(s) \sin(t-s)\, ds.$

(f) $f_6(t) = 8t^2 - 3\int_0^t f_6(s) \sin(t-s)\, ds.$

(g) $f_7(t) = t^2 - 2\int_0^t f_7(t-s) \sinh 2s ds.$

(h) $f_8(t) = 1 + 2\int_0^t f_8(t-s) \cos s ds.$

(i) $f_9(t) = 9e^{2t} - 2\int_0^t f_9(t-s) \cos s ds.$

18. Resolver los siguientes sistemas de ecuaciones utilizando la Transformada de Laplace

(a) $\begin{cases} 2x' + 2x + y' - y = 3t, \\ x' + x + y' + y = 1, \\ x(0) = 1, \\ y(0) = 3. \end{cases}$

(b) $\begin{cases} x' - 2x - y' - y = 6e^{3t}, \\ 2x' - 3x + y' - 3y = 6e^{3t}, \\ x(0) = 3, \\ y(0) = 0. \end{cases}$

19. Resolver los siguientes problemas de valor inicial

(a) $\begin{cases} y'' + 3ty' - 6y = 1, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(b) $\begin{cases} ty'' - 2y' + ty = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(c) $\begin{cases} ty'' - ty' + y = -4, \\ y(0) = 2, \\ y'(0) = -1. \end{cases}$

(d) $\begin{cases} y'' + ty' - y = 0, \\ y(0) = 0, \\ y'(0) = 3. \end{cases}$

(e) $\begin{cases} ty'' + (t-2)\, y' + y = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

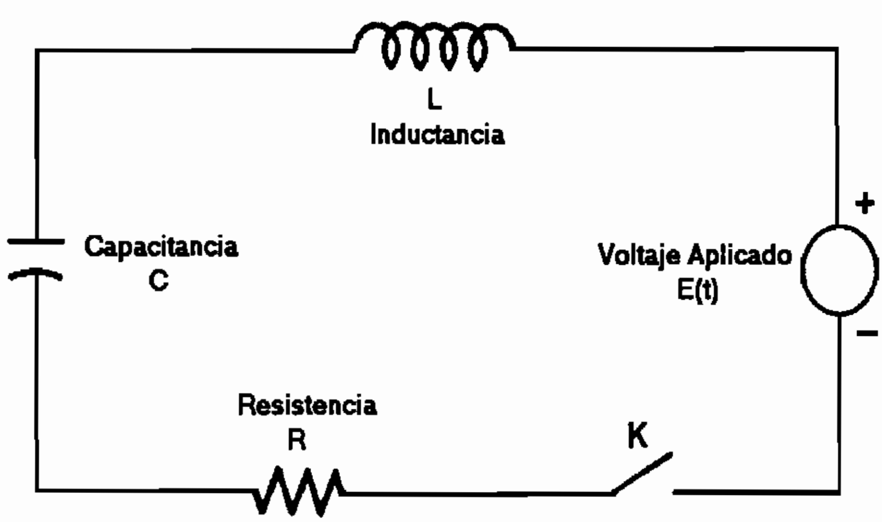
(f) $\begin{cases} ty'' + (3t-1)\, y' + 3y = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(g) $\begin{cases} ty'' - (4t+1)\, y' + 2\,(2t+1)\, y = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

(h) $\begin{cases} ty'' + 2\,(t-1)\, y' - 2y = 0, \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$

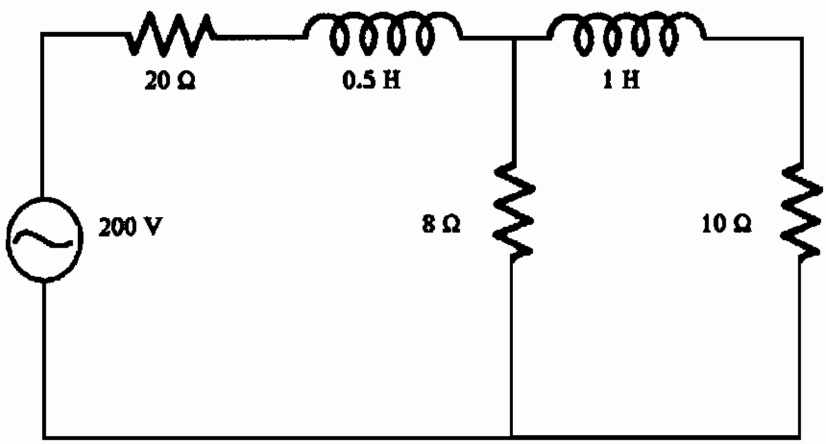
(i) $\begin{cases} ty'' - 2y + ty = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$

20. El circuito RLC de la siguiente figura



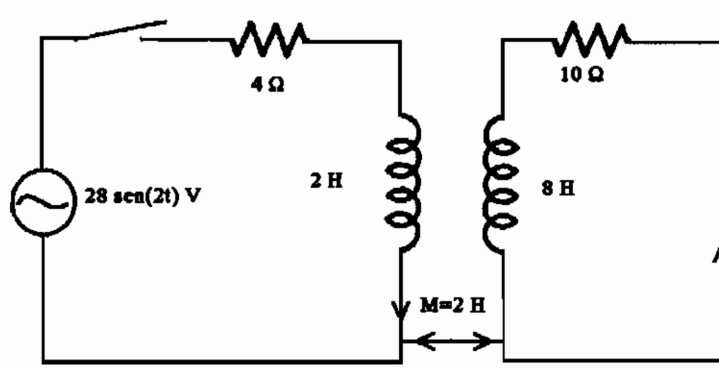
está formado por una resistencia R, un condensador C y un inductor L conectado en serie a una fuente de voltaje $v(t)$. Antes de cerrar el interruptor en el tiempo $t = 0$, tanto la carga en el condensador como la corriente resultante en el circuito son cero. Determinar la carga $q(t)$ en el condensador y la corriente resultante $i(t)$ en el circuito en el tiempo T sabiendo que $R = 160\Omega$, $L = 1H$, $c = 10^{-4}F$ y $v(t) = 20V$.

21. En la siguiente red en paralelo no hay flujo de corriente en ninguno de los lazos antes del cierre del interruptor en el tiempo $t = 0$.



Deducir las corrientes $i_1(t)$ e $i_2(t)$ que circulan en cada malla en el tiempo t.

22. Un voltaje $e(t)$ es aplicado a un primer circuito en el tiempo $t = 0$, y la inducción mutua M conduce la corriente $i_2(t)$ en el segundo circuito de la figura



Si previo al cierre del interruptor, las corrientes en ambos circuitos son cero, determinar la corriente inducida $i_2(t)$ en el segundo circuito en el tiempo t.

GRÁFICAS DE LOS EJERCICIOS DEL TEMA SERIES, INTEGRALES Y TRANSFORMADA DE FOURIER
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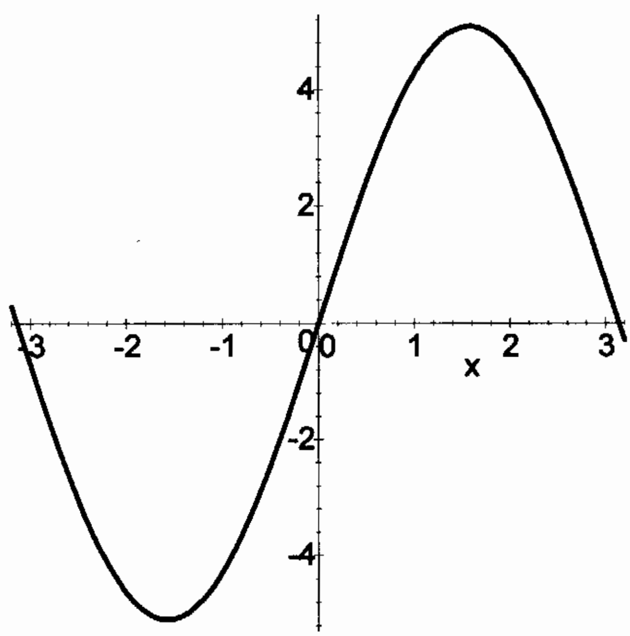
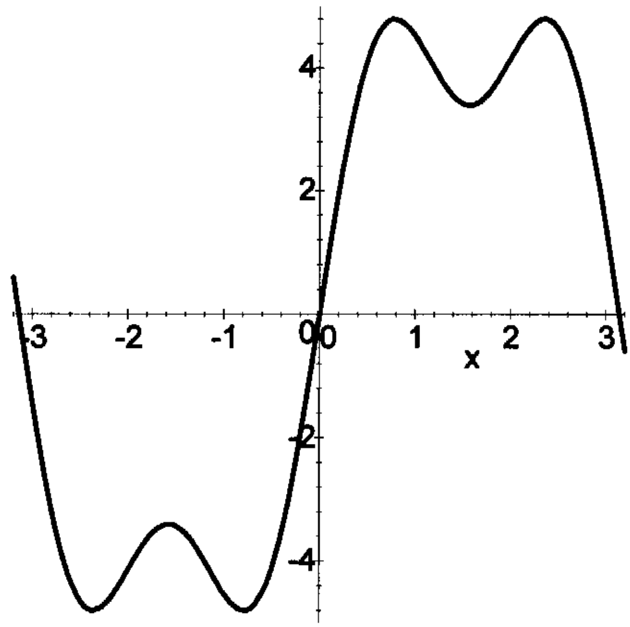
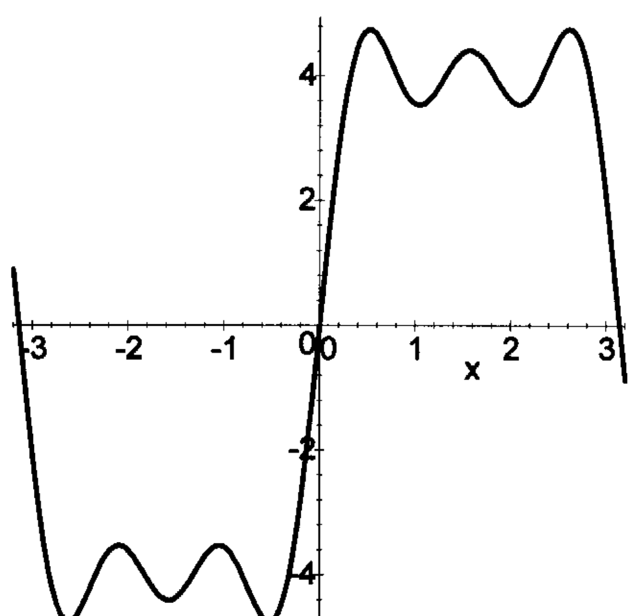
ONDA CUADRADA



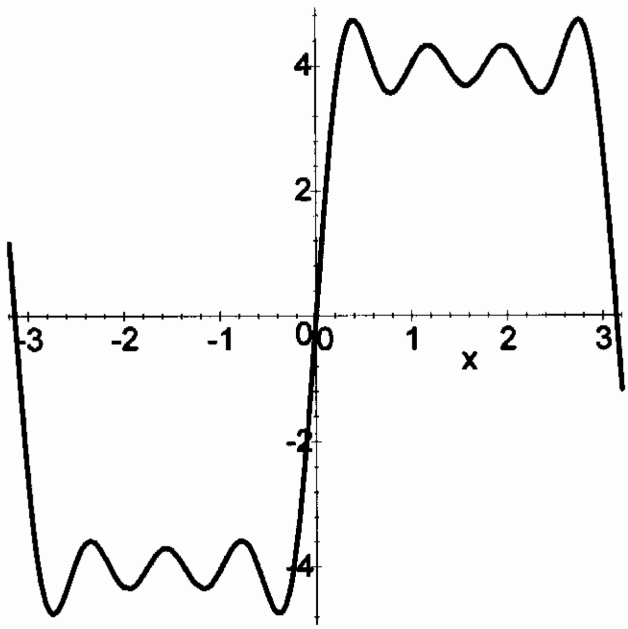
$$S_1 = \frac{4k}{\pi} \operatorname{sen} x$$



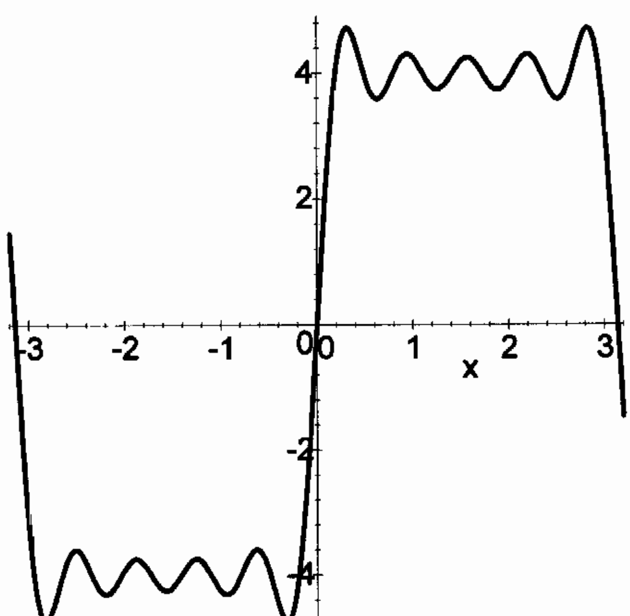
$$S_2 = \frac{16}{\pi}\left(\operatorname{sen} x + \frac{1}{3}\operatorname{sen} 3x\right)$$



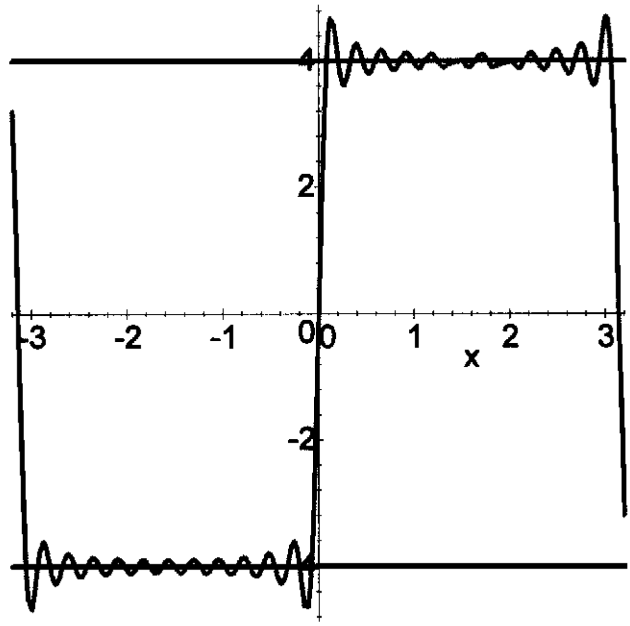
$$S_3 = \frac{16}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$$



$$S_4 = \frac{16}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x\right)$$



$$S_5 = \frac{16}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x\right)$$



S_{12}

EJERCICIOS RELACIÓN 6

TRANSFORMADA DE LAPLACE

1: Utilizar la identidad $\cos^2 \alpha = \frac{1}{2}(1+\cos 2\alpha)$ para obtener la transformada de Laplace de $\cos^2 kt$ y $\operatorname{sen}^2 kt$

<u>Solución</u>:

Por definición tenemos que:

$$\mathcal{L}[\cos^2 kt](z) = \int_0^{+\infty} e^{-zt} \cdot \cos^2 kt \, dt =$$

$$= \int_0^{+\infty} e^{-zt} \cdot \frac{1}{2}(1+\cos 2kt)\, dt = \frac{1}{2}\int_0^{+\infty} e^{-zt} dt +$$

$$+ \frac{1}{2}\int_0^{+\infty} e^{-zt} \cdot \cos 2kt \, dt$$

Se tiene que:

$$\int_0^{+\infty} e^{-zt} dt = \lim_{b\to+\infty} \int_0^{b} e^{-zt} dt = \lim_{b\to+\infty} \left. \frac{e^{-zt}}{-z} \right|_0^b = \frac{1}{z}$$

$$\int e^{-zt} dt \left\{ \begin{array}{l} u=-zt \\ du=-z\,dt \\ dt = \frac{-1}{z} du \end{array} \right\} = -\frac{1}{z}\int e^u du = \frac{-1}{z}\cdot e^u = -\frac{1}{z} e^{-zt}$$

$$\int_0^{+\infty} e^{-zt} \cdot \cos 2kt\, dt = \lim_{b\to\infty} \int_0^b e^{-zt} \cdot \cos 2kt\, dt =$$

$$= \lim_{b\to+\infty} \frac{2kz}{4k^2+z^2} \cdot e^{-zt}(\operatorname{sen} 2kt - \cos 2kt)\Big|_0^b = -\frac{2kz}{4k^2+z^2}$$

$$I = \int e^{-zt} \cdot \cos 2kt\, dt \quad (\text{por partes}) \quad \left\{ \begin{array}{l} u = e^{-zt} \Rightarrow du = -z \cdot e^{-zt} dt \\ dv = \cos 2kt\, dt \Rightarrow v = \frac{1}{2k} \cdot \operatorname{sen} 2kt \end{array} \right\}$$

$$I = \frac{1}{2k} \cdot \operatorname{sen} 2kt \cdot e^{-zt} + \frac{z}{2k} \int e^{-zt} \cdot \operatorname{sen} 2kt\, dt$$ (de nuevo aplicamos partes a esto)

$$\left\{ \begin{array}{l} u = e^{-zt} \Rightarrow du = -z e^{-zt} dt \\ dv = \operatorname{sen} 2kt\, dt \Rightarrow v = \frac{-1}{2k} \cdot \cos 2kt \end{array} \right\}$$

$$I = \frac{1}{2k} \operatorname{sen} 2kt \cdot e^{-zt} + \frac{z}{2k} \left(-\frac{1}{2k} \cdot \cos 2kt \cdot e^{-zt} - \frac{z}{2k} \int e^{-zt} \cos 2kt\, dt \right)$$

es decir:

$$I = \frac{z}{2k} \cdot e^{-zt} (\operatorname{sen} 2kt - \cos 2kt) - \frac{z^2}{4k^2} I$$

$$I\left(1 + \frac{z^2}{4k^2}\right) = \frac{z}{2k} e^{-zt} (\operatorname{sen} 2kt - \cos 2kt) \Rightarrow$$

$$\Rightarrow I = \int e^{-zt} \cos 2kt\, dt = \frac{2kz}{4k^2+z^2} e^{-zt} (\operatorname{sen} 2kt - \cos 2kt)$$

luego $\mathcal{L}[\cos^2 kt](z) = \frac{1}{2z} + \frac{1}{2}\left(-\frac{2kz}{4k^2+z^2}\right) = \frac{1}{2z} - \frac{kz}{4k^2+z^2}$

Al ser $\mathcal{L}[1](z) = \frac{1}{z}$, como $\operatorname{sen}^2 kt + \cos^2 kt = 1$ y teniendo en cuenta la linealidad de $\mathcal{L}$:

$$\mathcal{L}[1](z) = \mathcal{L}[\operatorname{sen}^2 kt + \cos^2 kt](z) = \mathcal{L}[\operatorname{sen}^2 kt](z) +$$

$$+ \mathcal{L}[\cos^2 kt](z) = \frac{1}{z} \Rightarrow \mathcal{L}[\operatorname{sen}^2 kt] = \frac{1}{z} - \mathcal{L}[\cos^2 kt](z)$$

$$\Rightarrow \mathcal{L}[\operatorname{sen}^2 kt](z) = \frac{1}{z} - \frac{1}{2z} + \frac{kz}{4k^2+z^2} \Rightarrow$$

$$\Rightarrow \mathcal{L}[\operatorname{sen}^2 kt](z) = \frac{1}{2z} + \frac{kz}{4k^2+z^2}$$

2) Determinar $\mathcal{L}[f(t)](z)$ donde

a) $f(t) = \begin{cases} 4 & \text{si } 0 \le t < 1 \\ 3 & \text{si } t \geq 1 \end{cases}$

Por definición

$$\mathcal{L}[f(t)](z) = \int_0^{+\infty} e^{-zt} \cdot f(t)\,dt = 4\int_0^1 e^{-zt} \cdot dt + 3\int_1^{+\infty} e^{-zt} dt =$$

$$= 4 \cdot \left[\frac{e^{-zt}}{-z} \right]_0^1 + 3 \lim_{b \to +\infty} \left[\frac{e^{-zt}}{-z} \right]_1^{b} =$$

$$= - \frac{4e^{-z} - 4}{z} + 3 \lim_{b \to +\infty} \left(\frac{e^{-bz}}{-z} + \frac{e^{-z}}{-z} \right) = - \frac{4e^{-z}}{z} + \frac{4}{z} + \frac{3e^{-z}}{z} =$$

$$\mathcal{L}[f(t)](z) = \frac{4 - e^{-z}}{z} \quad \text{si } \operatorname{Re} z > 0$$

Este ejercicio se podría haber hecho de forma más elegante y fácil utilizando la función escalón unidad de Heaviside y teniendo en cuenta que $\mathcal{L}[H(t-a)](z) = \dfrac{e^{-az}}{z}$. Veamos:

$$f(t) = 4[H(t) - H(t-1)] + 3H(t-1) = 4H(t) - H(t-1).$$

Por tanto, aprovechando la linealidad de $\mathcal{L}$, tenemos:

$$\mathcal{L}[f(t)](z) = \mathcal{L}[4H(t) - H(t-1)](z) = 4\mathcal{L}[H(t)](z) - \mathcal{L}[H(t-1)](z)$$

$$= 4 \cdot \frac{e^{-0 \cdot z}}{z} - \frac{e^{-1 \cdot z}}{z} = 4 \cdot \frac{1}{z} - \frac{e^{-z}}{z} = \frac{4 - e^{-z}}{z}$$

b) $f(t) = \begin{cases} 1 & \text{si } 0 \leq t \leq 2 \\ t & \text{si } t \geq 2 \end{cases}$

i) Sin utilizar la función de Heaviside

Por definición tenemos que

$$\mathcal{L}[f(t)](z) = \int_0^{+\infty} e^{-tz} \cdot f(t)\,dt = \int_0^2 e^{-tz}\,dt + \int_2^{+\infty} e^{-tz} \cdot t\,dt$$

$$\int_0^2 e^{-tz}\,dt = \left.\frac{e^{-tz}}{-z}\right|_0^2 = \frac{1}{z} - \frac{e^{-2z}}{z} = \frac{1-e^{-2z}}{z}$$

$$\int_2^{+\infty} t\cdot e^{-tz}\,dt = \lim_{b\to\infty}\int_2^b t\cdot e^{-tz}\,dt = -\frac{1}{z}e^{-tz}\left(1+\frac{1}{z}\right)\Big|_2^b =$$

$$\frac{1}{z}e^{-2z}\left(1+\frac{1}{z}\right) = \frac{z+1}{z^2}e^{-2z} \quad \text{Si } \operatorname{Re} z > 0$$

$$\int t\cdot e^{-tz}\,dt \text{ (por partes) } \left\{\begin{array}{l} u = t \Rightarrow du = dt \\ dv = e^{-tz}dt \Rightarrow v = -\frac{1}{z}e^{-tz} \end{array}\right\} =$$

$$= -\frac{1}{z}e^{-tz}\cdot t + \frac{1}{z}\int e^{-tz}\,dt = -\frac{1}{z}e^{-tz} - \frac{1}{z^2}\cdot e^{-tz} =$$

$$= \frac{-1}{z}e^{-tz}\left(1+\frac{1}{z}\right)$$

Luego:

$$\mathcal{L}[f(t)](z) = \frac{1-e^{-2z}}{z} + \frac{z+1}{z^2}\cdot e^{-2z} = \frac{1}{z} + \frac{1}{z}e^{-2z}\left(\frac{z+1}{z} - 1\right)$$

$$\mathcal{L}[f(t)](z) = \frac{1}{z}\left[1 + e^{-2z}\cdot\frac{1}{z}\right]$$

ii) Utilizando la función de Heaviside:

Se tiene que $f(t) = H(t) - H(t-2) + t\cdot H(t-2)$

Es decir que podemos poner:

$$f(t) = H(t) - H(t-2) + [(t-2)+2] \cdot H(t-2) =$$

$$f(t) = H(t) - H(t-2) + (t-2) \cdot H(t-2) + 2H(t-2)$$

$$f(t) = H(t) + H(t-2) + (t-2) \cdot H(t-2)$$, tendremos en cuenta que:

$$\mathcal{L}[H(t-a)] = \frac{e^{-az}}{z}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-az} \cdot \mathcal{L}[f(t)](z)$$

Por tanto:

$$\mathcal{L}[f(t)](z) = \mathcal{L}[H(t)](z) + \mathcal{L}[H(t-2)]^{(z)} + \mathcal{L}[(t-2)H(t-2)](z) =$$

$$= \frac{e^{-0t}}{z} + \frac{e^{-2t}}{z} \quad \mathcal{L}[t](z) = \frac{1}{z} + \frac{e^{-2t}}{z} + e^{-2t} \cdot \frac{1}{z^2}$$

$$\mathcal{L}[f(t)](z) = \frac{1}{z}\left[1 + e^{-2t}\left(1+\frac{1}{z}\right)\right] = \frac{1}{z}\left[1 + e^{-2t} \cdot \frac{z+1}{z}\right]$$

(en algo me he debido equivocar, no da el mismo resultado que antes, ¡revísalo si quieres!)

c) $$f(t) = \begin{cases} t & \text{si } 1 \le t \le 2 \\ 0 & \text{si } t \geq 2 \end{cases}$$

i) Sin la función de Heaviside

Por definición $\mathcal{L}[f(t)](z) = \int_0^{+\infty} e^{-zt} \cdot f(t)\, dt$

Como f ha de estar definida en $[0,+\infty[$, redefinimos f de la siguiente forma:

$$f(t)=\begin{cases} 0 & \text{si } 0\leq t<1 \\ t & \text{si } 1\leq t<2 \\ 0 & \text{si } t\geq 2 \end{cases}$$

Entonces:

$$\mathcal{L}[f(t)](z)=\int_1^2 e^{-tz}\cdot t\,dt=-\frac{1}{z}e^{-tz}\left(t+\frac{1}{z}\right)\Big|_1^2=$$

$$=-\frac{1}{z}e^{-2z}\left(2+\frac{1}{z}\right)+\frac{1}{z}e^{-z}\left(1+\frac{1}{z}\right)$$

$$\int t\cdot e^{-tz}dt \quad (\text{por partes}) \left\{\begin{array}{l} u=t \Rightarrow du=dt \\ dv=e^{-tz}dt \Rightarrow v=-\dfrac{e^{-tz}}{z} \end{array}\right\}=$$

$$=-\frac{t\,e^{-tz}}{z}+\frac{1}{z}\int e^{-tz}dt=\frac{-t\cdot e^{-tz}}{z}-\frac{1}{z^2}e^{-tz}=$$

$$=-\frac{1}{z}e^{-tz}\left(t+\frac{1}{z}\right)$$

ii) Utilizando la función de Heaviside.

De nuevo redefinimos la función de modo que esté definida en $[0,+\infty[$:

$$f(t)=\begin{cases} 0 & \text{si } 0\leq t<1 \\ t & \text{si } 1\leq t<2 \\ 0 & \text{si } t\geq 2 \end{cases}$$

Se verifica que $f(t) = t\,H(t-1) - t\,H(t-2) \Rightarrow$

$\Rightarrow f(t) = (t-1)\cdot H(t-1) + H(t-1) - (t-2)\,H(t-2) - 2H(t-2)$.

Aplicando la linealidad del operador $\mathcal{L}$:

$$\mathcal{L}[f(t)](z) = \mathcal{L}[(t-1)\cdot H(t-1)](z) + \mathcal{L}[H(t-1)](z) -$$

$$- \mathcal{L}[(t-2)H(t-2)](z) - 2\,\mathcal{L}[H(t-2)](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[f(t)](z) = e^{-z}\cdot \mathcal{L}[t](z) + \frac{e^{-z}}{z} - e^{-2z}\cdot \mathcal{L}[t](z) -$$

$$- 2\cdot \frac{e^{-2z}}{z} = e^{-z}\cdot \frac{1}{z^2} + \frac{e^{-z}}{z} - e^{-2z}\cdot \frac{1}{z^2} - 2\cdot \frac{e^{-z}}{z} =$$

$$= \frac{e^{-z}}{z}\left(1 + \frac{1}{z}\right) - \frac{e^{-2z}}{z}\left(2 + \frac{1}{z}\right)$$

d)
$$f(t) = \begin{cases} \operatorname{sen} 2t & \text{si } 0 < t < \pi \\ 0 & \text{si } t \geq \pi \end{cases}$$

Podemos poner $f(t) = \operatorname{sen} 2t\, H(t) - \operatorname{sen} 2t\, H(t-\pi) \Rightarrow$

$f(t) = \operatorname{sen} 2t \cdot H(t) + \operatorname{sen} 2(t-\pi)\cdot H(t-\pi)$ ya que para todo angulo α se verifica $\operatorname{sen}\alpha = -\operatorname{sen}(\alpha - \pi)$

Por tanto:

$$\mathcal{L}[f(t)](z) = \mathcal{L}[\operatorname{sen} 2t\cdot H(t)](z) + \mathcal{L}[\operatorname{sen} 2(t-\pi)\,H(t-\pi)](z)$$

$$\mathcal{L}[f(t)](z) = e^{-0z} \cdot \mathcal{L}[\operatorname{sen} 2t] + e^{-\pi z}\mathcal{L}[\operatorname{sen} 2t](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[f(t)](z) = \left(1 + e^{-\pi z}\right) \cdot \mathcal{L}[\operatorname{sen} 2t](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[f(t)](z) = (1 + e^{-\pi z}) \cdot \frac{-2}{4+z^2} = -2 \cdot \frac{1+e^{-\pi z}}{4+z^2}$$

$$\mathcal{L}[\operatorname{sen} 2t](z) = \int_0^{+\infty} e^{-tz} \cdot \operatorname{sen} 2t \, dt = \lim_{b\to+\infty} \int_0^b e^{-tz} \cdot \operatorname{sen} 2t \, dt =$$

$$= \lim_{b\to+\infty} \left. \frac{2e^{-tz}\left(\frac{z}{2}\operatorname{sen} 2t - \cos 2t\right)}{4+z^2} \right|_0^b = \frac{-2}{4+z^2}$$

$$I = \int e^{-tz} \cdot \operatorname{sen} 2t \, dt \text{ (por partes reiteradas)} \left\{ \begin{array}{l} u = e^{-tz} \Rightarrow du = -z \cdot e^{-tz} dt \\ dv = \operatorname{sen} 2t \, dt \Rightarrow v = \frac{-1}{2} \cos 2t \end{array} \right.$$

$$I = -\frac{1}{2} e^{-tz} \cdot \cos 2t - \frac{z}{2} \int e^{-tz} \cdot \cos 2t \, dt \quad \left\{ \begin{array}{l} u = e^{-tz} \Rightarrow du = -z e^{-tz} dt \\ dv = \cos 2t \, dt \Rightarrow v = \frac{1}{2} \operatorname{sen} 2t \end{array} \right.$$

$$I = -\frac{1}{2} e^{-tz} \cos 2t - \frac{z}{2}\left(\frac{1}{2} e^{-tz} \operatorname{sen} 2t + \frac{z}{2} I\right)$$

$$I\left(1 + \frac{z^2}{4}\right) = \frac{1}{2} e^{-tz}\left(\frac{z}{2} \operatorname{sen} 2t - \cos 2t\right)$$

$$I = \frac{2e^{-tz}\left(\frac{z}{2} \operatorname{sen} 2t - \cos 2t\right)}{4+z^2}$$

3.- La función de onda triangular, que denotamos por $T(t,c)$ viene dada por:

$$T(t,c) = \begin{cases} t & \text{si } 0 \leq t < c \\ 2c-t & \text{si } c \leq t < 2c \end{cases} \quad \text{y}$$

$$T(t+2c, c) = T(t,c)$$

Dibujar la función y determinar su transformada de Laplace

Solución:

La función T es periódica de periodo $2c$. Aplicamos el teorema para la transformada de Laplace de funciones periódicas:

Si $f \in \mathcal{E}_\gamma$ y f es periódica de periodo p, entonces:

$$\mathcal{L}[f(t)](z) = \frac{1}{1-e^{-pz}} \cdot \int_0^p e^{-zt} f(t)\,dt$$

Como $\lim\limits_{t \to \infty} \frac{T(t,c)}{e^t} = 0 \Rightarrow T$ es de orden exponencial de exponente $\gamma = 1$

Si aplicamos el teorema para la transformada de Laplace para funciones periódicas:

$$\mathcal{L}\left[T(t,c)\right](z) = \frac{1}{1-e^{-2cz}} \cdot \int_0^{2c} e^{-zt} \cdot f(t)\,dt \text{ , es decir:}$$

$$\mathcal{L}\left[T(t,c)\right](z) = \frac{1}{1-e^{-2cz}} \cdot \left[\int_0^{c} e^{-zt} \cdot t\,dt + \int_c^{2c} e^{-zt} \cdot (2c-t)\,dt \right]$$

Se tiene que:

$$\int e^{-zt} \cdot t\,dt \text{ (por partes) } \left\{ \begin{array}{l} u = t \Rightarrow du = dt \\ dv = e^{-zt}dt \Rightarrow v = -\dfrac{e^{-zt}}{z} \end{array} \right\} =$$

$$= -\frac{t}{z} e^{-zt} + \frac{1}{z} \cdot \int e^{-zt} dt = -\frac{e^{-zt}}{z} \left(t + \frac{1}{z} \right)$$

- $$\int_0^{c} e^{-zt} \cdot t\,dt = -\frac{e^{-cz}}{z} \left(c + \frac{1}{z} \right) + \frac{1}{z^2}$$

- $$\int_c^{2c} e^{-zt} (2c-t)\,dt = 2c \int_c^{2c} e^{-zt} dt - \int_c^{2c} e^{-zt} \cdot t\,dt =$$

$$= 2c \cdot \left(-\frac{1}{z} \cdot e^{-zt} \right)\Bigg|_c^{2c} + \frac{e^{-zt}}{z} \left(t + \frac{1}{z} \right)\Bigg|_c^{2c} =$$

$$= 2c \left(-\frac{1}{z} e^{-2cz} + \frac{1}{z} \cdot e^{-cz} \right) + \frac{e^{-2cz}}{z} \cdot \left(2c + \frac{1}{2c} \right) -$$

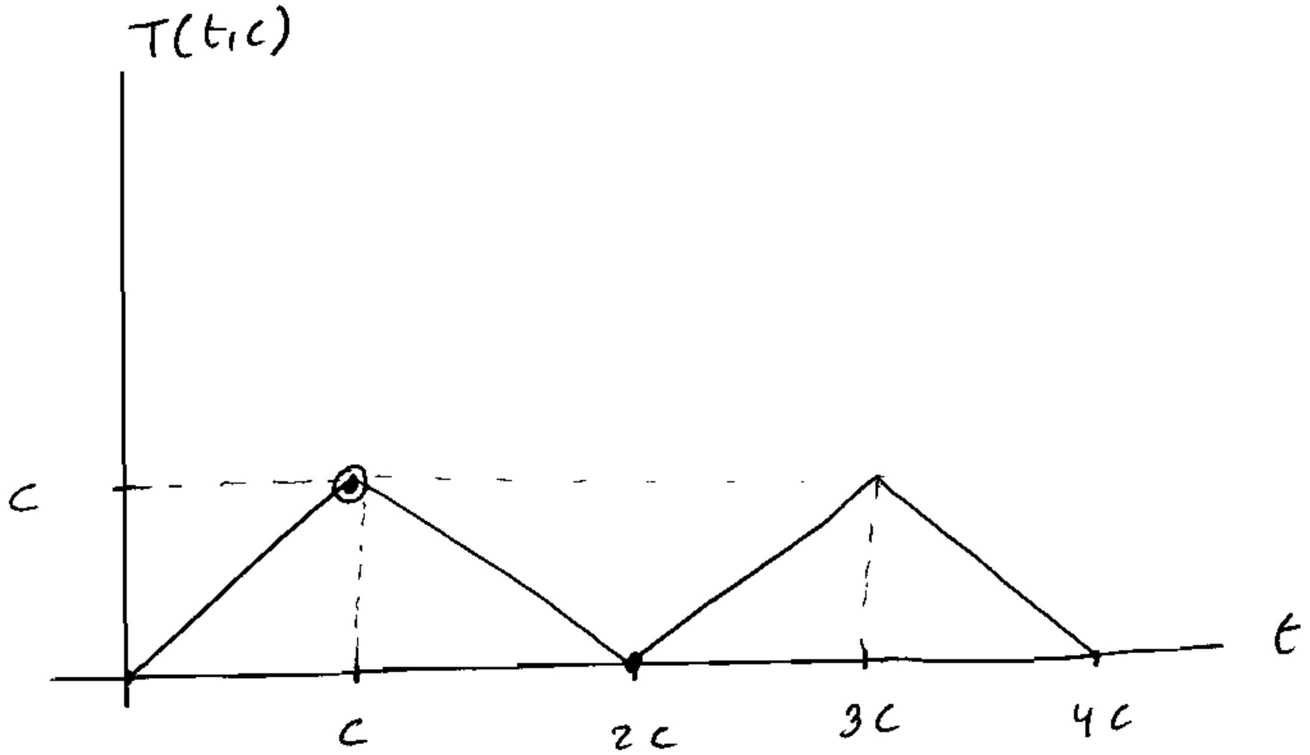
$$- \frac{e^{-cz}}{z} \left(c + \frac{1}{z} \right) = c \cdot \frac{1}{z} \cdot e^{-cz} + \frac{1}{2cz} \cdot e^{-2cz}$$

Luego:

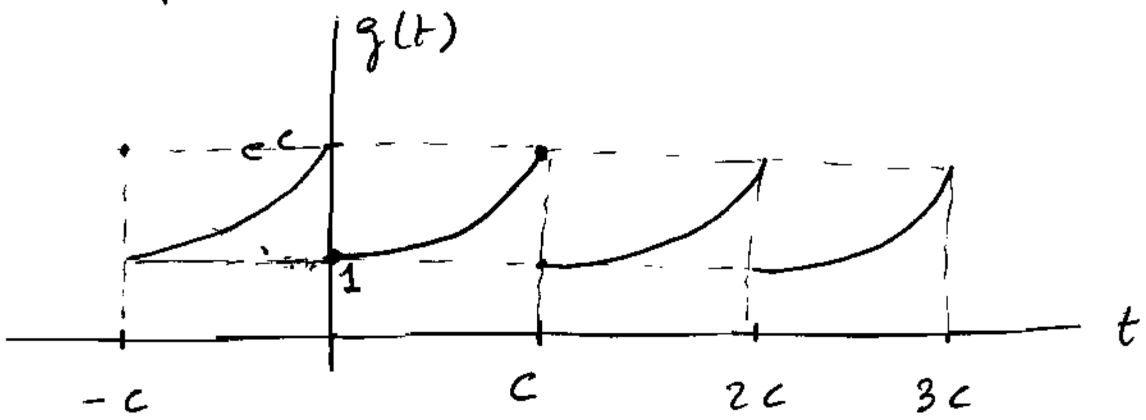
$$\mathcal{L}[T(t,c)](z) = \frac{1}{1-e^{-2cz}}\left[\cancel{-c\cdot\frac{1}{z}\cdot e^{-cz}} - \frac{1}{z^2}\cdot e^{-cz} + \frac{1}{z^2} \cancel{+c\cdot\frac{1}{z} e^{-cz}} + \frac{1}{2cz} e^{-2cz}\right]$$

$$\mathcal{L}[T(t,c)](z) = \frac{1}{1-e^{-2cz}}\left[\frac{1}{z^2}(1-e^{-cz}) + \frac{1}{2cz}\cdot e^{-2cz}\right]$$

Vamos a dibujar la función (¡¡ja, ja, ja!!)



4º Dibujar la gráfica de la función $g(t) = e^t$ siendo $0 \le t < c$ tal que $g(t+c) = g(t)$. Determinar su transformada de Laplace.



Evidentemente $g(t) = e^t$, $g(t+c) = g(t)$ es de orden exponencial. Por tanto podemos aplicar el teorema para la transformada de Laplace de una función periódica. $g(t)$ es periódica de periodo c.

Por tanto:

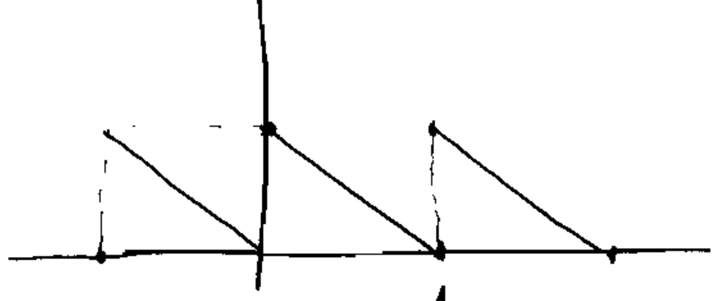
$$\mathcal{L}[g(t)](z) = \frac{1}{1-e^{-cz}} \cdot \int_0^c e^{-tz} \cdot e^t \, dz \Rightarrow$$

$$\mathcal{L}[g(t)](z) = \frac{1}{1-e^{-cz}} \cdot \int_0^c e^{t(1-z)} dz = \frac{1}{1-e^{-cz}} \cdot \left. \frac{e^{t(1-z)}}{1-z} \right|_0^c$$

$$\mathcal{L}[g(t)](z) = \frac{1}{1-e^{-cz}} \left(\frac{e^{c(1-z)}}{1-z} - \frac{1}{1-z} \right)$$

5º Dibujar la gráfica de la función $h(t) = 1-t$ siendo $0 \leq t < 1$ tal que $h(t+1) = h(t)$. Determinar su transformada de Laplace.

La función h es periódica de periodo 1.



La transformada de Laplace de h, ya que es de orden exponencial $\lim_{t\to\infty} \frac{h(t)}{e^t} = 0$, según el teorema para transformadas de funciones periódicas, vale

$$\mathcal{L}[h(t)](z) = \frac{1}{1-e^{-z}} \int_0^1 e^{-tz} \cdot (1-t)\, dt =$$

$$= \frac{1}{1-e^{-z}} \left(-\frac{2}{z} e^{-z} - \frac{1}{z^2} e^{-z} + \frac{1}{z} + \frac{1}{z^2} \right)$$

$$\int e^{-tz}(1-t)dt = \int e^{-tz} dt - \int e^{-tz} \cdot t\, dt = -\frac{1}{z} e^{-tz} -$$

$$- \frac{e^{-zt}}{z} \left(t + \frac{1}{z} \right)$$

$$\int_0^1 e^{-tz}(1-t)\, dt = -\frac{1}{z} e^{-tz} - \frac{1}{z} \cdot e^{-zt} \left(t + \frac{1}{z} \right) \Big|_0^1 =$$

$$= -\frac{1}{z} e^{-z} - \frac{1}{z} e^{-z} \left(1 + \frac{1}{z} \right) + \frac{1}{z} + \frac{1}{z^2} = -\frac{2}{z} e^{-z} - \frac{1}{z^2} e^{-z} + \frac{1}{z} + \frac{1}{z^2}$$

6.- Determinar la transformada inversa de Laplace de las siguientes funciones:

a) $F_1(z) = \dfrac{1}{z^2+2z+10}$

Aplicaremos el teorema para el cálculo de la transformada inversa que dice:

$\mathcal{L}^{-1}[F_1(z)](t) = \sum \text{Res}(e^{zt} \cdot F(z), z_i)$ siendo z_i polos de $F(z)$.

$$z^2+2z+10=0 \Rightarrow z = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} \begin{cases} -1+3i \\ -1-3i \end{cases}$$

$$\mathcal{L}^{-1}[F_1(z)](t) = \text{Res}\left[\frac{e^{zt}}{z^2+2z+10}, -1+3i\right] + \text{Res}\left[\frac{e^{zt}}{z^2+2z+10}, -1-3i\right]$$

$p(z) = e^{zt} \Rightarrow p(-1+3i) = e^{(-1+3i)t}$; $p(-1-3i) = e^{-(1+3i)t}$

$q(z) = z^2+2z+10 \Rightarrow q'(z) = 2z+2 \Rightarrow q'(-1+3i) = 6i$; $q'(-1-3i) = -6i$

$$\text{Res}\left[\frac{e^{zt}}{z^2+2z+10}, -1+3i\right] = \frac{e^{(-1+3i)t}}{6i}$$

$$\text{Res}\left[\frac{e^{zt}}{z^2+2z+10}, -1-3i\right] = -\frac{e^{-(1+3i)t}}{6i}$$. Entonces:

$$\mathcal{L}^{-1}[F_1(z)](t) = \frac{e^{(-1+3i)t} - e^{-(1+3i)t}}{6i}$$

b) $F_2(z) = \dfrac{1}{z^2-4z+8}$

$$z^2-4z+8=0 \Rightarrow z = \frac{4 \pm \sqrt{16-32}}{2} = \frac{+4 \pm 4i}{2} \left\langle \begin{array}{l} 2+2i \\ 2-2i \end{array} \right.$$

Luego:

$$\mathcal{L}^{-1}[F_2(z)](t) = \operatorname{Res}\left[\frac{e^{tz}}{z^2-4z+8}, 2+2i\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2-4z+8}, 2-2i\right]$$

$$p(z) = e^{tz} \Rightarrow p(2+2i) = e^{(2+2i)t} \Rightarrow p(2-2i) = e^{(2-2i)t}$$

$$q(z) = z^2-4z+8 \Rightarrow q'(z) = 2z-4\ ;\ q'(2+2i) = 4+8i\ ;\ q'(2-2i) = 4-4i$$

$$\operatorname{Res}\left[\frac{e^{tz}}{z^2-4z+8}, 2+2i\right] = \frac{e^{(2+2i)t}}{4+8i}$$

$$\operatorname{Res}\left[\frac{e^{tz}}{z^2-4z+8}, 2-2i\right] = \frac{e^{(2-2i)t}}{4-4i}$$

Entonces:

$$\mathcal{L}^{-1}[F_2(z)](t) = \frac{e^{(2+2i)t}}{4+8i} + \frac{e^{(2-2i)t}}{4-4i}$$

Los apartados c), d) y e) son iguales, no los hago, te los dejo para que practiques.

f) $F_6(z) = \dfrac{z}{z^2+4z+4}$

$$z^2+4z+4=0 \Rightarrow z = -\frac{4 \mp \sqrt{16-16}}{2} \left\langle \begin{array}{l} -2 \\ -2 \end{array} \right.$$

$z=-2$ es un polo doble de $F_6(z) = \dfrac{z}{z^2+4z+4}$

Por tanto:

$$\mathcal{L}^{-1}[F_6(z)](t) = \operatorname{Res}\left[\frac{e^{tz}\cdot z}{z^2+4z+4}, -2\right]$$

$$\operatorname{Res}\left[\frac{e^{tz}\cdot z}{z^2+4z+4}, -2\right] = \operatorname{Res}\left[\frac{z\cdot e^{tz}}{(z+2)^2}, -2\right] =$$

$$= \lim_{z\to -2} \frac{d}{dz}\left[\cancel{(z+2)^2}\cdot \frac{z\cdot e^{tz}}{\cancel{(z+2)^2}}\right] = \lim_{z\to -2}\left(z\cdot t\cdot e^{tz} + e^{tz}\right) =$$

$$= -2t\cdot e^{-2t} + e^{-2t} = e^{-2t}(1-2t)$$

Luego $\mathcal{L}^{-1}[F_6(z)](t) = e^{-2t}(1-2t)$

Me salto los apartados g) y h)

i) $F_9(z) = \dfrac{2z+3}{(z+4)^3}$. $z=-4$ es un polo triple de $F_9(z)$

Por tanto:

$$\mathcal{L}^{-1}[F_9(z)](t) = \operatorname{Res}\left[\frac{(2z+3)\,e^{tz}}{(z+4)^3}, -4\right] =$$

$$= \lim_{z\to -4} \frac{1}{2!}\frac{d^2}{dz^2}\left[\cancel{(z+4)^3}\cdot \frac{(2z+3)\,e^{tz}}{\cancel{(z+4)^3}}\right] =$$

$$= \lim_{z\to -4} \frac{1}{2}\cdot(2+3t+2tz)e^{tz} = \frac{1}{2}(2+3t-8t)\,e^{-4t} =$$

$$= \frac{1}{2}(2-5t)\,e^{-4t}.$$

j) $F_{10}(z) = \dfrac{z^2}{(z-1)^4}$ tiene un polo de orden 4

en $z=1$.

$$\mathcal{L}^{-1}[F_{10}(z)](t) = \operatorname{Res}\left[\frac{e^{tz}\cdot z^2}{(z-1)^4}, 1\right] =$$

$$= \lim_{z\to 1} \frac{1}{3!} \frac{d^3}{dz^3}\left((z-1)^4 \cdot \frac{z^2 e^{tz}}{(z-1)^4}\right) =$$

$$= \lim_{z\to 1} \frac{1}{6} \cdot e^{tz}(t^3 z^2 + 6t^2 z + 4t + 2) = \frac{1}{6} e^t (t^3 + 6t^2 + 4t + 2)$$

7) Determinar la transformada inversa de Laplace de las siguientes funciones donde $a^2 \neq b^2$ y $ab \neq 0$.

a) $G_1(z) = \dfrac{1}{z^2 + az}$

Descomponemos la fracción racional propia en fracciones simples

$$\frac{1}{z^2+az} = \frac{1}{z(z+a)} = \frac{A}{z} + \frac{B}{z+a} = \frac{(A+B)z + Aa}{z(z+a)}$$

$$\left.\begin{array}{l} A+B=0 \\ aA = 1 \end{array}\right\} \begin{array}{l} B = -\dfrac{1}{a} \\ A = \dfrac{1}{a} \end{array}$$

Luego: $G_1(z) = \frac{1}{a} \cdot \frac{1}{z} - \frac{1}{a} \frac{1}{z+a}$ y por tanto:

$$\mathcal{L}^{-1}[G_1(z)](t) = \frac{1}{a} \mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - \frac{1}{a} \mathcal{L}^{-1}\left[\frac{1}{z+a}\right](t)$$

$$\mathcal{L}^{-1}[G_1(z)](t) = \frac{1}{a} - \frac{1}{a} e^{-at} = \frac{1}{a}\left(1 - e^{-at}\right)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z}, 0\right] = \frac{p(0)}{q'(0)} = 1$$

$p(z) = e^{tz} \to p(0) = 1$

$q(z) = z \Rightarrow q'(z) = 1 \Rightarrow q'(0) = 1$

$$\mathcal{L}^{-1}\left[\frac{1}{z+a}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z+a}, -a\right] = \frac{p(-a)}{q'(-a)} = e^{-at}$$

$p(z) = e^{tz} \Rightarrow p(-a) = e^{-at}$

$q(z) = z+a \Rightarrow q'(z) = 1 \Rightarrow q'(-a) = 1$

b) $G_2(z) = \dfrac{2z^2-1}{z(z+1)^2}$

En primer lugar descomponemos en fracciones simples:

$$\frac{2z^2-1}{z(z+1)^2} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{(z+1)^2} = \frac{A(z+1)^2 + Bz(z+1) + Cz}{z(z+1)^2}$$

$Az^2 + 2Az + A + Bz^2 + Bz + Cz = 2z^2 - 1 \Rightarrow (A+B)z^2 + (2A+B+C)z + A = 2z^2 - 1$

$$\left.\begin{array}{l} A+B=2 \\ 2A+B+C=0 \\ A=-1 \end{array}\right\} \quad A=-1 \Rightarrow B=3 \Rightarrow C=-1$$

Entonces :

$$\frac{2z^2-1}{z(z+1)^2} = \frac{-1}{z} + \frac{3}{z+1} - \frac{1}{(z+1)^2}.$$ Por tanto :

$$\mathcal{L}^{-1}\left[\frac{2z^2-1}{z(z+1)^2}\right](t) = -\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) + 3\mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = \operatorname{Res}\left(\frac{e^{tz}}{z}, 0\right) = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

$p(t) = e^{tz} \Rightarrow p(0) = 1$

$q(z) = z \Rightarrow q'(z) = 1 \Rightarrow q'(0) = 1.$

$$\mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z+1}, -1\right] = e^{-t}$$

$p(z) = e^{tz} \Rightarrow p(-1) = e^{-t}$

$q(z) = z+1 \Rightarrow q'(z) = 1 \Rightarrow q'(-1) = 1$

$$\mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{(z+1)^2}, -1\right] = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{e^{tz}}{(z+1)^2}\right] =$$

$$= \lim_{z\to -1} t e^{tz} = t e^{-t}$$. Entonces :

$$\mathcal{L}^{-1}\left[\frac{2z^2-1}{z(z+1)^2}\right](t) = -1 + 3e^{-t} - te^{-t} = (3-t)e^{-t} - 1$$

c) $G_3(z) = \dfrac{2z^2+5z-4}{z^3+z^2-2z}$

$$z^3+z^2-2z=0 \Rightarrow z(z^2+z-2)=0 \begin{cases} z=0 \\ z=\dfrac{-1\pm\sqrt{1+8}}{2} = \begin{cases} 1 \\ -2 \end{cases} \end{cases}$$

$$\frac{2z^2+5z-4}{z(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2} = \frac{A(z-1)(z+2)+Bz(z+2)+Cz(z+1)}{z(z+1)(z+2)}$$

$$A(z^2+z-2)+B(z^2+2z)+C(z^2+z) = 2z^2+5z-4$$

$$\left.\begin{array}{l} A+B+C=2 \\ A+2B+C=5 \\ -2A=-4 \end{array}\right\} \begin{cases} \left.\begin{array}{l} B+C=0 \\ 2B+C=3 \end{array}\right\} \longrightarrow B=3 \Rightarrow C=-3 \\ A=2 \end{cases}$$

Luego:

$G_3(z) = \dfrac{2}{z} + \dfrac{3}{z-1} - \dfrac{3}{z+2}$, entonces:

$$\mathcal{Z}^{-1}[G_3(z)](t) = 2\,\mathcal{Z}^{-1}\left[\frac{1}{z}\right](t) + 3\,\mathcal{Z}^{-1}\left[\frac{1}{z-1}\right](t) - 3\,\mathcal{Z}^{-1}\left[\frac{1}{z+2}\right](t)$$

$\mathcal{Z}^{-1}\left[\dfrac{1}{z}\right](t) = 1$

$\mathcal{Z}^{-1}\left[\dfrac{1}{z-1}\right](t) = e^{t}$

$\mathcal{Z}^{-1}\left[\dfrac{1}{z+2}\right](t) = e^{-2t}$. Por tanto:

$$\mathcal{Z}^{-1}[G_3(z)](t) = 2+3e^{t}-3e^{-2t}$$

d) $G_4(z) = \dfrac{4z+4}{z^2(z-2)}$

$$\frac{4z+4}{z^2(z-2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-2} = \frac{Az(z-2)+B(z-2)+Cz^2}{z^2(z-2)}$$

$$Az^2 - 2Az + Bz - 2B + Cz^2 = 4z+4$$

$$\left.\begin{aligned} A+C&=0 \\ -2A+B&=4 \\ -2B&=4 \end{aligned}\right\} \quad \begin{aligned} &C=3 \\ &A=\frac{B-4}{2}=-3 \\ &B=-2 \end{aligned}$$

$$G_4(z) = \frac{-3}{z} - \frac{2}{z^2} + \frac{3}{z-2}$$

$$\mathcal{L}^{-1}[G_4(z)](t) = -3\,\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - 2\,\mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) + 3\,\mathcal{L}^{-1}\left[\frac{1}{z-2}\right](t)$$

$$\mathcal{L}^{-1}[G_4(z)](t) = -3 - 2t + 3e^{2t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z^2}, 0\right] = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \cdot \frac{e^{tz}}{z^2}\right] =$$

$$= \lim_{z\to 0} t\cdot e^{tz} = t$$

$$\mathcal{L}^{-1}\left[\frac{1}{z-2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z-2}, 2\right] = e^{2t}$$

e) $G_5(z) = \dfrac{5z-2}{z^2(z+2)(z-1)}$

$$\frac{5z-2}{z^2(z+2)(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+2} + \frac{D}{z-1}$$

$$\frac{5z-2}{z^2(z+2)(z-1)} = \frac{Az(z+2)(z-1) + B(z+2)(z-1) + Cz^2(z-1) + Dz^2(z+2)}{z^2(z+2)(z-1)}$$

$$A(z^3+z^2-2z) + B(z^2+z-2) + C(z^3-z^2) + D(z^3+2z^2) = 5z-2$$

$$A + C + D = 0$$

$$A + B - C + 2D = 0$$

$$-2A = 5 \quad \Rightarrow \quad A = -\frac{5}{2}$$

$$-2B = -2 \quad \Rightarrow \quad B = 1$$

$$\left.\begin{array}{l} C + D = \frac{5}{2} \\ -C + 2D = \frac{3}{2} \end{array}\right\} \Rightarrow 3D = \frac{8}{2} = 4$$

$$D = \frac{4}{3}$$

$$C = \frac{5}{2} - \frac{4}{3} = \frac{7}{6}$$

Luego:

$$G_5(z) = -\frac{5}{2} \cdot \frac{1}{z} + \frac{1}{z^2} + \frac{7}{6} \cdot \frac{1}{z+2} + \frac{4}{3} \frac{1}{z-1}$$

$$\mathcal{Z}^{-1}[G_5(z)](t) = -\frac{1}{5} \mathcal{Z}^{-1}\left[\frac{1}{z}\right](t) + \mathcal{Z}^{-1}\left[\frac{1}{z^2}\right](t) + \frac{7}{6} \mathcal{Z}^{-1}\left[\frac{1}{z+2}\right](t)$$

$$+ \frac{4}{3} \mathcal{Z}^{-1}\left[\frac{1}{z-1}\right](t) = -\frac{1}{5} + t + \frac{7}{6} e^{-2t} + \frac{4}{3} e^{t}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{z}\right](t) = 1 \qquad \mathcal{Z}^{-1}\left[\frac{1}{z+2}\right](t) = e^{-2t}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{z^2}\right](t) = t \qquad \mathcal{Z}^{-1}\left[\frac{1}{z-1}\right](t) = e^{t}$$

f) $G_6(z) = \dfrac{1}{z^3(z^2+1)}$

$$\frac{1}{z^3(z^2+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{Mz+N}{z^2+1}$$

$$Az^2(z^2+1) + Bz(z^2+1) + C(z^2+1) + z^3(Mz+N) = 1$$

$$\left.\begin{array}{l} A+M=0 \\ B+N=0 \\ A+C=0 \\ B=0 \\ C=1 \end{array}\right\} B=0,\ C=1 \Rightarrow A=-1 \Rightarrow N=0 \Rightarrow M=1$$

$G_6(z) = \dfrac{-1}{z} + \dfrac{1}{z^3} + \dfrac{z}{z^2+1}$. Por tanto:

$$\mathcal{L}^{-1}[G_6(z)](t) = -\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{z^3}\right] + \mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^3}\right](t) = \operatorname{Res}\left[\frac{e^{zt}}{z^3}, 0\right] = \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[e^{zt}\right] =$$

$$= \lim_{z\to 0} t^2 e^{zt} = t^2$$

$$\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t) = \operatorname{Res}\left[\frac{z\cdot e^{zt}}{z^2+1}, +i\right] + \operatorname{Res}\left[\frac{z\cdot e^{zt}}{z^2+1}, -i\right] =$$

$$= \frac{e^{it}}{2} + \frac{e^{-it}}{2} = \frac{e^{it}+e^{-it}}{2} = \cos t$$

Luego:

$$\mathcal{L}^{-1}\left[G_6(z)\right](t) = -1+t^2+\cos t$$

g) $G_7(z) = \dfrac{2z-3}{z^2-4z+8}$ (creo que se ha equivocado, lo ha copiado de un libro inglés)

$$z^2-4z+8=0 \Rightarrow z = \frac{4\pm\sqrt{16-32}}{2} = \frac{4\pm 4i}{2} \begin{cases} 2+2i \\ 2-2i \end{cases}$$

Los puntos singulares son $z_0 = 2+2i$, $z_1 = 2-2i$ que a su vez son polos de orden 1.

$$\mathcal{L}^{-1}\left[G_7(z)\right](t) = \mathcal{L}^{-1}\left[\frac{2z-3}{z^2-4z+8}\right](t) = \operatorname{Res}\left[\frac{(2z-3)e^{tz}}{z^2-4z+8}, 2+2i\right] +$$

$$+ \operatorname{Res}\left[\frac{(2z-3)e^{tz}}{z^2-4z+8}, 2-2i\right] = \frac{(1+4i)e^{(2+2i)t} - (1-4i)e^{(2-2i)t}}{4i}$$

$$\operatorname{Res}\left[\frac{(2z-3)e^{tz}}{z^2-4z+8}, 2+2i\right] = \frac{p(2+2i)}{q'(2+2i)} = \frac{(1+4i)e^{(2+2i)t}}{4i}$$

$q'(z) = 2z-4$

$$\operatorname{Res}\left[\frac{(2z-3)e^{tz}}{z^2-4z+8}, 2-2i\right] = \frac{p(2-2i)}{q'(2-2i)} = \frac{(1-4i)\cdot e^{(2-2i)t}}{-4i}$$

h) $G_8(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$

Lo primero descomponemos en fracciones simples:

$$\frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{Mz+N}{z^2+a^2} + \frac{kz+\alpha}{z^2+b^2}$$

$$(z^2+b^2)(Mz+N)+(z^2+a^2)(kz+\alpha)=1.$$

$$Mz^3+Nz^2+Mb^2z+Nb^2+kz^3+\alpha z^2+ka^2z+\alpha a^2=1$$

$$\begin{array}{l} M+K=0 \\ N+\alpha=0 \\ Mb^2+Ka^2=0 \\ Nb^2+\alpha a^2=1 \end{array} \left\{ \begin{array}{l} K=-M \\ \alpha=-N \\ Mb^2-Ma^2=0 \Rightarrow M(b^2-a^2)=0 \Rightarrow M=0 \text{ ya} \\ \text{que } a^2\neq b^2 \Rightarrow a^2-b^2\neq 0 \Rightarrow K=0 \end{array} \right.$$

$$Nb^2-Na^2=1 \Rightarrow N(b^2-a^2)=1 \Rightarrow N=\frac{1}{b^2-a^2} \Rightarrow$$

$$\Rightarrow \alpha = \frac{1}{a^2-b^2}$$

Luego:

$$\frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{b^2-a^2}\cdot\frac{1}{z^2+a^2} + \frac{1}{a^2-b^2}\cdot\frac{1}{z^2+b^2} = G_8(z)$$

Por tanto

$$\mathcal{L}^{-1}[G_8(z)](t) = \frac{1}{b^2-a^2}\cdot\mathcal{L}^{-1}\left[\frac{1}{z^2+a^2}\right](t) + \frac{1}{a^2-b^2}\,\mathcal{L}^{-1}\left[\frac{1}{z^2+b^2}\right](t)$$

Tenemos que:

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+a^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z^2+a^2}, ai\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2+a^2}, -ai\right] =$$

$$= \frac{e^{tai}}{2ai} - \frac{e^{-tai}}{2ai} = \frac{1}{a}\,\frac{e^{ati}-e^{-ati}}{2i} = \frac{1}{a}\operatorname{sen} at$$

De igual modo:

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+b^2}\right](t) = \frac{1}{b}\operatorname{sen} bt$$

Por lo tanto:

$$\mathcal{L}^{-1}\left[G_8(z)\right] = \frac{1}{a(b^2-a^2)}\operatorname{sen} at + \frac{1}{b(a^2-b^2)}\operatorname{sen} bt$$

i) $$G_9(z) = \frac{z}{(z^2+a^2)(z^2+b^2)} = \frac{Mz+N}{z^2+a^2} + \frac{kz+\alpha}{z^2+b^2}$$

$$(Mz+N)(z^2+b^2) + (kz+\alpha)(z^2+a^2) = z$$
$$(M+k)z^3 + (N+\alpha)z^2 + (Mb^2+ka^2)z + Nb^2+\alpha a^2 = z$$

$$\left.\begin{array}{l} M+k=0 \\ N+\alpha=0 \\ Mb^2+ka^2=1 \\ Nb^2+\alpha a^2=0 \end{array}\right\} \begin{array}{l} k=-M \\ \alpha=-N \\ Mb^2-Ma^2=1 \Rightarrow M(b^2-a^2)=1 \Rightarrow M=\frac{1}{b^2-a^2} \\ \text{ya que } b^2-a^2\neq 0 \end{array}$$

$$Nb^2-Na^2=0 \Rightarrow N(b^2-a^2)=0 \Rightarrow N=0 \text{ ya que } b^2-a^2\neq 0$$

$$M=\frac{1}{b^2-a^2} \Rightarrow k=\frac{1}{a^2-b^2} \qquad N=0 \Rightarrow \alpha=0.$$

Entonces:

$$G_9(z) = \frac{1}{b^2-a^2}\cdot\frac{z}{z^2+a^2} + \frac{1}{a^2-b^2}\cdot\frac{z}{z^2+b^2}.$$ Por lo tanto:

$$\mathcal{L}^{-1}\left[G_9(z)\right](t) = \frac{1}{b^2-a^2}\,\mathcal{L}^{-1}\left[\frac{z}{z^2+a^2}\right](t) + \frac{1}{a^2-b^2}\,\mathcal{L}^{-1}\left[\frac{z}{z^2+b^2}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{z}{z^2+a^2}\right](t) = \operatorname{Res}\left[\frac{z\cdot e^{tz}}{z^2+a^2}, ai\right] + \operatorname{Res}\left[\frac{z\,e^{tz}}{z^2+a^2}, -ai\right]$$

$$= \frac{\cancel{ai}\cdot e^{ati}}{2\cancel{ai}} + \frac{\cancel{ai}\,e^{-ati}}{2\cancel{ai}} = \cos at$$

De la misma manera:

$\mathcal{L}^{-1}\left[\frac{z}{z^2+a^2}\right](t) = \cos bt.$ Luego:

$$\mathcal{L}^{-1}\left[G_9(z)\right](t) = \frac{1}{b^2-a^2}\cos at + \frac{1}{a^2-b^2}\cos bt$$

j) $$G_{10}(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)} = \frac{Mz+N}{z^2+a^2} + \frac{Kz+\alpha}{z^2+b^2}$$

$$(M+K)z^3 + (N+\alpha)z^2 + (Mb^2+Ka^2)z + Nb^2 + \alpha a^2 = z^2$$

$$\begin{array}{l} M+K = 0 \\ N+\alpha = 1 \\ Mb^2+Ka^2 = 0 \\ Nb^2+\alpha a^2 = 0 \end{array}
\left\{\begin{array}{l} K = -M \\ \alpha = 1-N \\ M(b^2-a^2) = 0 \Rightarrow M=0 \text{ ya que } b^2-a^2 \neq 0 \\ Nb^2 + a^2 - a^2N = 0;\ N(b^2-a^2) = -a^2 \end{array}\right.$$

y como $b^2-a^2 \neq 0 \Rightarrow N = \dfrac{a^2}{a^2-b^2} \Rightarrow \alpha = 1 - \dfrac{a^2}{a^2-b^2} = \dfrac{-b^2}{a^2-b^2}$

$$M = 0 \Rightarrow K = 0;\quad N = \frac{a^2}{a^2-b^2} \quad \alpha = \frac{b^2}{b^2-a^2}$$

Luego:

$$G_{10}(z) = \frac{a^2}{a^2-b^2} \cdot \frac{1}{z^2+a^2} + \frac{b^2}{b^2-a^2} \cdot \frac{1}{z^2+b^2}$$ Por tanto:

$$\mathcal{L}^{-1}[G_{10}(z)](t) = \frac{a^2}{a^2-b^2} \mathcal{L}^{-1}\left[\frac{1}{z^2+a^2}\right](t) + \frac{b^2}{b^2-a^2} \mathcal{L}^{-1}\left[\frac{1}{z^2+b^2}\right](t)$$

$$\Rightarrow \mathcal{L}^{-1}[G_{10}(z)](t) = \frac{a^2}{a^2-b^2} \cdot \cos at + \frac{b^2}{b^2-a^2} \cdot \cos bt$$

8º) Resolver cada uno de los siguientes problemas de valor inicial por medio del método de la transformada de Laplace. Verificar la solución.

a) $\begin{cases} y' = e^t \\ y(0) = 2 \end{cases}$

Si aplicamos la transformada de Laplace a la ecuación $y' = e^t \Rightarrow \mathcal{L}[y'](z) = \mathcal{L}[e^t](z) \Rightarrow$

$z \cdot \mathcal{L}[y](z) - y(0) = \mathcal{L}[e^t](z)$. Como $\mathcal{L}[e^t](z) = \frac{1}{z-1}$

$$z \cdot Y(z) - 2 = \frac{1}{z-1} \Rightarrow z \cdot Y(z) = \frac{1}{z-1} + 2 = \frac{2z-1}{z-1}$$

$$\Rightarrow Y(z) = \frac{2z-1}{z(z-1)} \Rightarrow y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{2z-1}{z(z-1)}\right](t)$$

$$\frac{2z-1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{Az-A+Bz}{z(z-1)}$$

$$\left.\begin{array}{l} A+B=2 \\ -A=-1 \end{array}\right\} A=1 \Rightarrow B=1$$

$$\frac{2z-1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{z}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{z-1}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{z-1}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z-1}, 1\right] = \frac{e^t}{1} = e^t$$

La solución de la ecuación diferencial es:

$$y(t) = 1 + e^t.$$

b) $$\left.\begin{array}{l} y' - y = e^{-t} \\ y(0) = 1 \end{array}\right\}$$

Aplicamos la transformada de Laplace a $y' - y = e^{-t}$

$$\Rightarrow \mathcal{L}[y'](z) - \mathcal{L}[y](z) = \mathcal{L}[e^{-t}]$$

$$z \cdot \mathcal{L}[y](z) - y(0) - \mathcal{L}[y](z) = \mathcal{L}[e^{-t}]$$

Como $\mathcal{L}[e^{-t}](z) = \frac{1}{z+1}$, llamado $Y(z) = \mathcal{L}[y](z)$

nos queda que:

$$z\,Y(z) - 1 - Y(z) = \frac{1}{z+1}$$

$$Y(z)\cdot(z-1) = \frac{1}{z+1} + 1 \Rightarrow Y(z) = \frac{z+2}{(z-1)(z+1)}$$. Aplicando la transformada inversa:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{z+2}{(z-1)(z+1)}\right](t)$$

Vamos a descomponer en fracciones simples:

$$\frac{z+2}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1} \Rightarrow Az + A + Bz - B = z+2 \Rightarrow$$

$$\Rightarrow (A+B)z + (A-B) = z+2$$

$$\begin{matrix} A+B = 1 \\ A-B = 2 \end{matrix} \Rightarrow A = \frac{3}{2} \Rightarrow B = -\frac{1}{2}$$

$$\frac{3z+2}{(z-1)(z+1)} = \frac{3}{2}\cdot\frac{1}{z-1} - \frac{1}{2}\cdot\frac{1}{z+1}.$$

$$\mathcal{L}^{-1}\left[\frac{3z+2}{(z-1)(z+1)}\right](t) = \frac{3}{2}\,\mathcal{L}^{-1}\left[\frac{1}{z-1}\right](t) - \frac{1}{2}\,\mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z-1}\right] = e^{t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z+1}\right] = e^{-t}$$

La solución de la e.d.o. es:

$$y(t) = \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}$$

c) $\begin{cases} y'' + a^2 y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

Aplicamos la transformada de Laplace a $y''+a^2y=0$

$$\mathcal{L}[y''+a^2y](z) = \mathcal{L}[0](z)$$

$$\mathcal{L}[y''] + a^2\mathcal{L}[y] = 0$$

$$z^2\mathcal{L}[y](z) - z\,y(0) - y'(0) + a^2\mathcal{L}[y](z) = 0$$

Haciendo $Y(z) = \mathcal{L}[y](z)$ nos queda:

$$z^2 Y(z) - z + a^2 Y(z) = 0$$

$Y(z)(z^2+a^2) = z \Rightarrow Y(z) = \dfrac{z}{z^2+a^2}$. Aplicando la transformada de Laplace inversa:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{z}{z^2+a^2}\right] = \operatorname{Res}\left[\frac{e^{tz}z}{z^2+a^2}, ai\right]$$

$$+ \operatorname{Res}\left[\frac{e^{tz}z}{z^2+a^2}, -ai\right] = \frac{e^{ati}}{2} + \frac{e^{-ati}}{2} = \cos at$$

$p(z) = e^{tz}\,z \begin{cases} p(ai) = ai\,e^{ati} \\ p(-ai) = -ai\cdot e^{-ati} \end{cases}$

$q(z) = z^2+a^2 \Rightarrow q'(z) = 2z \begin{cases} q'(ai) = 2ai \\ q'(-ai) = -2ai \end{cases}$

La solución es $y(t) = \cos at$. Vamos a verificarlo:

$$\left.\begin{array}{l} y(t) = \cos at \\ y'(t) = -a\,\text{sen}\,at \\ y''(t) = -a^2\cos at \end{array}\right\} \begin{array}{l} y'' + a^2 y = 0 \ \text{. Sustituyendo:} \\ -a^2\cos at + a^2\cos at = 0 \quad \text{cierto} \end{array}$$

Veamos si se cumplen las condiciones iniciales:

$$y(0) = \cos a\cdot 0 = \cos 0 = 1$$

$$y'(0) = -a\,\text{sen}\,a\cdot 0 = -a\,\text{sen}\,0 = 0$$

d) $$\left.\begin{array}{l} y'' - 3y' + 2y = e^{3t} \\ y(0) = 0 \\ y'(0) = 0 \end{array}\right\}$$

Aplicamos la transformada de Laplace a la ecuación:

$$\mathcal{L}[y'' - 3y' + 2y](z) = \mathcal{L}[e^{3t}](z)$$

$$\mathcal{L}[y''] - 3\mathcal{L}[y'](z) + 2\mathcal{L}[y](z) = \frac{1}{z-3}$$

$$z^2\mathcal{L}[y](z) - zy(0) - y'(0) - 3z\mathcal{L}[y] - 3y(0) + 2\mathcal{L}[y](z) = \frac{1}{z-3}$$

Haciendo $Y(z) = \mathcal{L}[y](z)$ nos queda:

$$z^2 Y(z) - 3zY(z) + 2Y(z) = \frac{1}{z-3}$$

$$Y(z)\left(z^2 - 3z + 2\right) = \frac{1}{z-3} \Rightarrow Y(z) = \frac{1}{(z^2-3z+2)(z-3)}$$

Es decir que:

$$Y(z) = \frac{1}{(z-1)(z-2)(z-3)}$$. Descomponemos en fracciones simples:

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2) = 1$$

$$A(z^2-5z+6) + B(z^2-4z+3) + C(z^2-3z+2) = 1$$

$$\left.\begin{array}{l} A+B+C=0 \\ -5A-4B-3C=0 \\ 6A+3B+2C=1 \end{array}\right\} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -5 & -4 & -3 & 0 \\ 6 & 3 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right) \qquad \left.\begin{array}{r} A+B+C=0 \\ B+2C=0 \\ 2C=1 \end{array}\right\} \begin{array}{l} A=\frac{1}{2} \\ B=-1 \\ C=\frac{1}{2} \end{array}$$

Luego:

$$Y(z) = \frac{1}{2}\cdot\frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{2}\,\frac{1}{z-3}$$. Aplicando la transformada de Laplace inversa:

$$\mathcal{L}^{-1}[Y(z)](t) = \frac{1}{2}\,\mathcal{L}^{-1}\left[\frac{1}{z-1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{z-2}\right](t) +$$

$$+\frac{1}{2}\,\mathcal{L}^{-1}\left[\frac{1}{z-3}\right](t) \Rightarrow \boxed{y(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}}$$

$\mathcal{L}^{-1}\left[\frac{1}{z-1}\right](t) = e^{+t}$; $\mathcal{L}^{-1}\left[\frac{1}{z-2}\right] = e^{2t}$; $\mathcal{L}^{-1}\left[\frac{1}{z-3}\right] = e^{3t}$

Vamos a verificar la solución $y(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}$ para la ecuación $y'' - 3y' + 2y = e^{3t}$

$y'(t) = \frac{1}{2}e^{t} - 2e^{2t} + \frac{3}{2}e^{3t}$

$y''(t) = \frac{1}{2}e^{t} - 4e^{2t} + \frac{9}{2}e^{3t}$. Sustituyendo en la ecuación queda:

$$\cancel{\frac{1}{2}e^{t}} \cancel{- 4e^{2t}} \cancel{+ \frac{9}{2}e^{3t}} \cancel{- \frac{3}{2}e^{t}} \cancel{+ 6e^{2t}} \cancel{- \frac{9}{2}e^{3t}} \cancel{+ e^{t}} \cancel{- 2e^{2t}} + e^{3t} =$$

$= e^{3t}$ cierto. Veamos que se cumplen las condiciones iniciales:

$y(0) = \frac{1}{2}e^{0} - e^{0} + \frac{1}{2}e^{0} = \frac{1}{2} - 1 + \frac{1}{2} = 0$

$y'(0) = \frac{1}{2}e^{0} - 2e^{0} + \frac{3}{2}e^{0} = \frac{1}{2} - 2 + \frac{3}{2} = \frac{4}{2} - 2 = 0.$

e) $\left.\begin{array}{l} y'' + y = e^{-t} \\ y(0) = 0 \\ y'(0) = 0 \end{array}\right\}$

Aplicamos la transformada de Laplace a $y'' + y = e^{-t}$

Nos queda que:

$$\mathcal{L}[y''](z)+\mathcal{L}[y](z) = \mathcal{L}[e^{-t}](z)$$

$$z^2\mathcal{L}[y](z) - zy(0) - y'(0) + \mathcal{L}[y](z) = \frac{1}{z+1}$$

Haciendo $Y(z) = \mathcal{L}[y](z)$ obtenemos:

$$z^2 Y(z) + Y(z) = \frac{1}{z+1}$$

$$Y(z)\left(z^2+1\right) = \frac{1}{z+1} \Rightarrow Y(z) = \frac{1}{(z+1)(z^2+1)}$$

Descomponemos en fracciones simples:

$$\frac{1}{(z+1)(z^2+1)} = \frac{A}{z+1} + \frac{Bz+C}{z^2+1}$$

$$A(z^2+1) + (Bz+C)(z+1) = 1$$

$$Az^2+A+Bz^2+Bz+Cz+C = 1$$

$$\left.\begin{array}{l} A+B=0 \\ B+C=0 \\ A+C=1 \end{array}\right\} \quad \left.\begin{array}{l} A+B=0 \\ A-B=1 \end{array}\right\} \quad 2A=1 \Rightarrow A=\frac{1}{2} \Rightarrow B=-\frac{1}{2} \Rightarrow C=\frac{1}{2}$$

Luego:

$$\frac{1}{(z+1)(z^2+1)} = \frac{1}{2}\,\frac{1}{z+1} + \frac{-\frac{1}{2}z+\frac{1}{2}}{z^2+1} = \frac{1}{2}\left(\frac{1}{z+1} + \frac{-z+1}{z^2+1}\right)$$

Como $Y(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{z-1}{z^2+1}\right]$, aplicando la transformada de Laplace inversa:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t) \frac{1}{2}\mathcal{L}^{-1}\left[\frac{z-1}{z^2+1}\right](t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t) = e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{z-1}{z^2+1}\right](t) = \operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, i\right] + \operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, -i\right]$$

$= \cos t - \operatorname{sen}(t)$ (está explicado a continuación)

$p(z) = e^{tz}(z-1) \Rightarrow p(i) = e^{ti}(i-1) \quad p(-i) = e^{-ti}(-i-1)$

$q(z) = z^2+1 \Rightarrow q'(z) = 2z \Rightarrow q'(i) = 2i;\ q'(-i) = -2i$

$$\operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, i\right] = \frac{e^{ti}(i-1)}{2i} = \frac{e^{ti}(-1-i)}{-2} = \frac{e^{ti}(1+i)}{2}$$

$$\operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, -i\right] = \frac{e^{-ti}(-i-1)}{-2i} = \frac{e^{-ti}(1-i)}{2}$$

Entonces:

$$\operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, i\right] + \operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, -i\right] = \frac{e^{ti}+e^{-ti}}{2} + i\cdot\frac{e^{ti}-e^{-ti}}{2}$$

$= \cos t - \frac{e^{ti}-e^{-ti}}{2i} = \cos t - \operatorname{sen} t$. La solución de la ecuación diferencial es $\boxed{y(t) = \frac{1}{2}\left(e^{-t} + \cos t - \operatorname{sen} t\right)}$

Vamos a verificarla

$$y(t) = \frac{1}{2}\left(e^{-t} + \cos t - \operatorname{sen} t\right)$$

$$y'' + y = e^{-t}$$

$$y' = \frac{1}{2}\left(-e^{-t} - \operatorname{sen} t - \cos t\right) = -\frac{1}{2}\left(e^{-t} + \operatorname{sen} t + \cos t\right)$$

$$y'' = -\frac{1}{2}\left(-e^{-t} + \cos t - \operatorname{sen} t\right)$$; sustituyendo:

$$-\frac{1}{2}\left(-e^{-t} + \cos t - \operatorname{sen} t\right) + \frac{1}{2}\left(e^{-t} + \cos t - \operatorname{sen} t\right) =$$

$$= \frac{1}{2}e^{-t} \cancel{- \frac{1}{2}\cos t} \cancel{+ \frac{1}{2}\operatorname{sen} t} + \frac{1}{2}e^{-t} \cancel{+ \frac{1}{2}\cos t} \cancel{- \frac{1}{2}\operatorname{sen} t} =$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}e^{-t} = e^{-t}$$ cierto.

Veamos que se verifican las condiciones iniciales

$y(0) = 0$
$y'(0) = 0$

$y(0) = \frac{1}{2}\left(e^{0} + \cos 0 - \operatorname{sen} 0\right) = \frac{1}{2}\left(1 + 1 - 0\right) = 1$ (no se cumple en algo he debido equivocarme, repásalo)

$y'(0) = -\frac{1}{2}\left(e^{0} + \operatorname{sen} 0 + \cos 0\right) = -\frac{1}{2} \cdot 2 = -1$ (de nuevo me he debido equivocar, repásalo si quieres).

Los apartados f, g, h me los salto, son similares a los que ya hemos hecho.

i) $\begin{cases} y''+3y'+2y=4t^2 \\ y(0)=0 \\ y'(0)=0 \end{cases}$

Aplicamos la transformada de Laplace a la ecuación diferencial:

$$y''+3y'+2y=4t^2 \Rightarrow \mathcal{L}[y''](z)+3\mathcal{L}[y'](z)+2\mathcal{L}[y](z)=4\mathcal{L}[t^2](z)$$

$$z^2\mathcal{L}[y](z)-zy(0)-y'(0)+3\Big(z\mathcal{L}[y](z)-y(0)\Big)+2\mathcal{L}[y](z)=$$

$= 4\cdot\dfrac{2!}{z^3}$ (aplicar que $\mathcal{L}[t^n](z)=\dfrac{n!}{z^{n+1}}$). Poniendo

$Y(z)=\mathcal{L}[y](z)$ nos queda:

$$z^2Y(z)+3zY(z)+2Y(z)=\frac{8}{z^3}$$

$$Y(z)\left(z^2+3z+2\right)=\frac{8}{z^3} \Rightarrow Y(z)=\frac{8}{z^3(z^2+3z+2)}$$

$Y(z)=\dfrac{8}{z^3(z+2)(z+1)}$. Vamos a descomponer en fracciones simples:

$$\frac{8}{z^3(z+1)(z+2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1} + \frac{E}{z+2}$$

$$A\, z^2(z+1)(z+2) + B z(z+1)(z+2) + C(z+1)(z+2) + D z^3(z+2) + E z^3(z+1) = 8$$

$$Az^4 + 3Az^3 + 2Az^2 + Bz^3 + 3Bz^2 + 2Bz + Cz^2 + 3Cz + 2C + Dz^4 + 2Dz^3 + Ez^4 + Ez^3 = 8$$

$$\left.\begin{aligned} A+D+E &= 0 \\ 3A+B+2D+E &= 0 \\ 2A+3B+C &= 0 \\ 2B+3C &= 0 \\ 2C &= 8 \end{aligned}\right\} \quad \begin{aligned} & 7+D+E=0 \\ & 21-6+2D+E=0 \\ & 2A-18+4=0 \Rightarrow A=7 \\ & B=-6 \\ & C=4 \end{aligned} \quad \left\{\begin{aligned} D+E &= -7 \\ 2D+E &= -15 \end{aligned}\right\} \; -D=+8$$

$$D=-8$$
$$E=1$$

Luego:

$$Y(z) = 7\cdot\frac{1}{z} - 6\cdot\frac{1}{z^2} + 4\cdot\frac{1}{z^3} - 8\cdot\frac{1}{z+1} + \quad \cdot \frac{1}{z+2},$$

Aplicando la transformada inversa:

$$y(t) = \mathcal{Z}^{-1}[Y(z)](t) = 7\cdot\mathcal{Z}^{-1}\left[\frac{1}{z}\right](t) - 6\cdot\mathcal{Z}^{-1}\left[\frac{1}{z^2}\right](t) + 4\cdot\mathcal{Z}^{-1}\left[\frac{1}{z^3}\right](t)$$

$$+ 6\cdot\mathcal{Z}^{-1}\left[\frac{1}{z+1}\right](t) - 27\cdot\mathcal{Z}^{-1}\left[\frac{1}{z+2}\right](t);$$

Según la propiedad $\mathcal{Z}[e^{wt}](z) = \frac{1}{z-w}$ tenemos que:

$$\mathcal{Z}^{-1}\left[\frac{1}{z}\right](t) = e^0 = 1$$

$$\mathcal{Z}^{-1}\left[\frac{1}{z+1}\right] = e^{-t}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{z+2}\right] = e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) = \text{Res}\left[\frac{e^{tz}}{z^2}, 0\right] = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \frac{e^{tz}}{z^2}\right] =$$

$$= \lim_{z\to 0} t\, e^{tz} = t$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^3}\right](t) = \text{Res}\left[\frac{e^{tz}}{z^3}, 0\right] = \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[z^3 \cdot \frac{e^{tz}}{z^3}\right] =$$

$$= \frac{1}{2} \lim_{z\to 0} t^2 e^{tz} = \frac{t^2}{2}$$

La solución de la ecuación diferencial anterior es:

$$\boxed{y(t) = 7 - 6t + 2t^2 - 8e^{-t} + e^{-2t}}$$

Vamos a comprobar la solución

$$y(0) = 7 - 0 + 0 - 8 + 1 = 0 \quad \text{cierto}$$

$$y'(t) = -6 + 4t + 8e^{-t} - 2 \cdot e^{-2t}$$

$$y'(0) = -6 + 8 - 2 = 0 \quad \text{cierto}$$

$$y'' = 4 - 8e^{-t} + 4e^{-2t}$$

$$y'' + 3y' + 2y = \cancel{4} \cancel{-8e^{-t}} \cancel{+4e^{-2t}} \cancel{-18} \cancel{+12t} \cancel{+24e^{-t}} \cancel{-6e^{-2t}} +$$

$$+ \cancel{14} \cancel{-12t} + 4t^2 \cancel{-16e^{-t}} \cancel{+2e^{-2t}} = 4t^2 \quad \text{cierto.}$$

j) $\begin{cases} \frac{1}{4} y'' - y' + y = \cos 2t \\ y(0) = 2 \\ y'(0) = 5 \end{cases}$

Aplicamos la transformada de Laplace a la ecuación diferencial:

$$\frac{1}{4} \mathcal{L}[y''] - \mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[\cos 2t](z)$$

$$\frac{1}{4}\left[z^2 \mathcal{L}[y](z) - z\,y(0) - y'(0) \right] - \left[z\mathcal{L}[y](z) - y(0)\right] +$$

$$+ \mathcal{L}[y](z) = \mathcal{L}[\cos 2t](z).$$

Si hacemos $Y(z) = \mathcal{L}[y](z)$, teniendo en cuenta que:

$$\mathcal{L}[\cos 2t](z) = \int_0^{+\infty} e^{-tz} \cos 2t \, dt = \lim_{b\to\infty} \int_0^b e^{-tz} \cos 2t \, dt =$$

$$= \lim_{b\to\infty} \left[e^{-bz} \frac{-z\cos 2b + 2\,\mathrm{sen}\, 2b}{z^2-4} + \frac{z}{z^2-4} \right] = \frac{z}{z^2-4} \quad \text{si } \operatorname{Re} z > 0$$

$I = \int \cos 2t \cdot e^{-tz} dz$ (por partes reiteradas) $\begin{cases} u = \cos 2t; \, du = -2\,\mathrm{sen}\, 2t\, dt \\ dv = e^{-tz} dt; \, v = -\dfrac{e^{-tz}}{z} \end{cases}$

$$I = -\frac{1}{z}\, e^{-tz} \cos 2t - \frac{2}{z}\int e^{-tz} \cdot \operatorname{sen} 2t\, dt \quad \begin{cases} u = \operatorname{sen} 2t;\ du = 2\cos 2t\, dt \\ dv = e^{-tz} dz \Rightarrow v = -\dfrac{e^{-tz}}{z} \end{cases}$$

$$I = -\frac{1}{z}\, e^{-tz}\cos 2t - \frac{2}{z}\left[-\frac{1}{z}\, e^{-tz} \operatorname{sen} 2t + \frac{2}{z}\int e^{-tz}\cos 2t\, dt\right]$$

$$I = -\frac{1}{z}\, e^{-tz}\cos 2t + \frac{2}{z^2}\, e^{-tz}\operatorname{sen} 2t - \frac{4}{z^2}\, I$$

$$z^2 I - 4I = -z \cdot e^{-tz}\cos 2t + 2\, e^{-tz}\operatorname{sen} 2t$$

$$I(z^2-4) = e^{-tz} \cdot (-z\cos 2t + 2\operatorname{sen} 2t)$$

$$I = \int e^{-zt} \cdot \cos 2t\, dt = e^{-tz} \cdot \frac{-z\cos 2t + 2\operatorname{sen} 2t}{z^2-4}$$

$$\int_0^b e^{-tz}\cos 2t\, dt = e^{-bz}\, \frac{-z\cos 2b + 2\operatorname{sen} 2b}{z^2-4} + \frac{z}{z^2-4}$$

La ecuación diferencial transformada es:

$$\frac{1}{4}\left[z^2 Y(z) - 2z - 5\right] - \left[z Y(z) - 5\right] + Y(z) = \frac{z}{z^2-4}$$

$$z^2 Y(z) - 2z - 5 - 4z Y(z) + 20 + 4 Y(z) = \frac{4z}{z^2-4}$$

$$z^2 Y(z) - 4z\, Y(z) + 4 Y(z) = \frac{4z}{z^2-4} + 2z - 15$$

$$Y(z)\left[z^2-4z+4\right] = \frac{4z+2z^3-8z-15z^2+60}{z^2-4}$$

$$Y(z) = \frac{2z^3-15z^2-4z+60}{(z^2-4)(z^2-4z+4)}$$

$$Y(z) = \frac{2z^3-15z^2-4z+60}{(z+2)(z-2)(z-2)^2}$$

Descomponemos en fracciones simples:

$$\frac{2z^3-15z^2-4z+60}{(z+2)(z-2)^3} = \frac{A}{z+2} + \frac{B}{z-2} + \frac{C}{(z-2)^2} + \frac{D}{(z-2)^3}$$

$$A(z-2)^3 + B(z-2)^2(z+2) + C(z-2)(z+2) + D(z+2) = 2z^3-15z^2-4z+60$$

$$A(z^3-6z^2-12z-8) + B(z^3+2z^2-4z-8) + C(z^2-4) + D(z+2) = 2z^3-15z^2-4z+60$$

$$\left.\begin{array}{l} A+B=2 \\ -6A+2B+C=-15 \\ -12A-4B+D=-4 \\ -8A-8B-4C+2D=60 \end{array}\right\} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -6 & 2 & 1 & 0 & -15 \\ -12 & -4 & 0 & 1 & -4 \\ -8 & -8 & -4 & 2 & 60 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 8 & 1 & 0 & -3 \\ 0 & 8 & 0 & 1 & 20 \\ 0 & 0 & -4 & 2 & 76 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 8 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 & -23 \\ 0 & 0 & -4 & 2 & 76 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 8 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 & -23 \\ 0 & 0 & 0 & -2 & -16 \end{array}\right) \quad \begin{array}{l} A+B=2 \\ 8B+C=-3 \\ C-D=-23 \\ -2D=-16 \end{array} \left\{\begin{array}{l} A=\frac{1}{2} \\ 8B=12\ ;\ B=\frac{3}{2} \\ C=-15 \\ D=8 \end{array}\right.$$

Luego:

$$Y(z) = \frac{1}{2} \cdot \frac{1}{z+2} + \frac{3}{2} \cdot \frac{1}{z-2} - 15 \cdot \frac{1}{(z-2)^2} + 8 \cdot \frac{1}{(z-2)^3}$$

Aplicando la transformada inversa:

$$\mathcal{L}^{-1}[Y(z)](t) = \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{z+2}\right](t) + \frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{z-2}\right](t) -$$

$$-15 \mathcal{L}^{-1}\left[\frac{1}{(z-2)^2}\right](t) + 8 \mathcal{L}^{-1}\left[\frac{1}{(z-2)^3}\right](t).$$

Aplicando que $\mathcal{L}[e^{wt}](z) = \frac{1}{z-w}$ obtenemos que:

$$\mathcal{L}^{-1}\left[\frac{1}{z+2}\right](t) = e^{-2t} \quad ; \quad \mathcal{L}^{-1}\left[\frac{1}{z-2}\right](t) = e^{2t}$$

Por otro lado obtenemos que:

- $\mathcal{L}^{-1}\left[\frac{1}{(z-2)^2}\right](t) = \mathrm{Res}\left[\frac{e^{tz}}{(z-2)^2}, 2\right] = \lim_{z\to 2} \frac{d}{dz}\left[(z-2)^2 \frac{e^{tz}}{(z-2)^2}\right] =$

$$= \lim_{z\to 2} t e^{tz} = t \cdot e^{2t}$$

- $\mathcal{L}^{-1}\left[\frac{1}{(z-2)^3}\right](t) = \mathrm{Res}\left[\frac{e^{tz}}{(z-2)^3}, 2\right] = \lim_{z\to 2} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-2)^3 \frac{e^{tz}}{(z-2)^3}\right] =$

$$= \lim_{z\to 2} \left[t^2 e^{tz}\right] = t^2 \cdot e^{2t}$$. Luego la solución es:

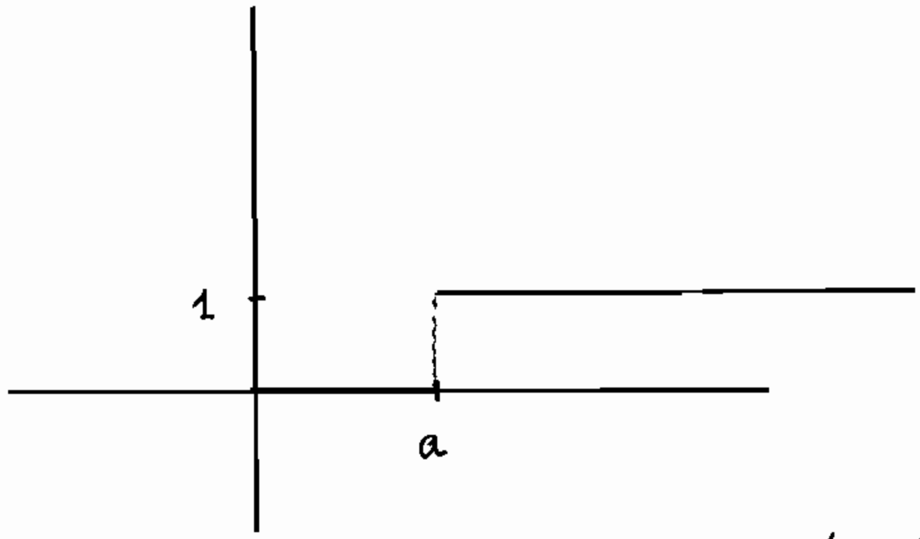
$$\mathcal{L}^{-1}[Y(z)](t) = y(t) = \frac{1}{2} e^{-2t} + \frac{3}{2} e^{2t} - 15t \cdot e^{2t} + 8t^2 e^{2t}$$

8') Representar la gráfica de las siguientes funciones para $t \geq 0$

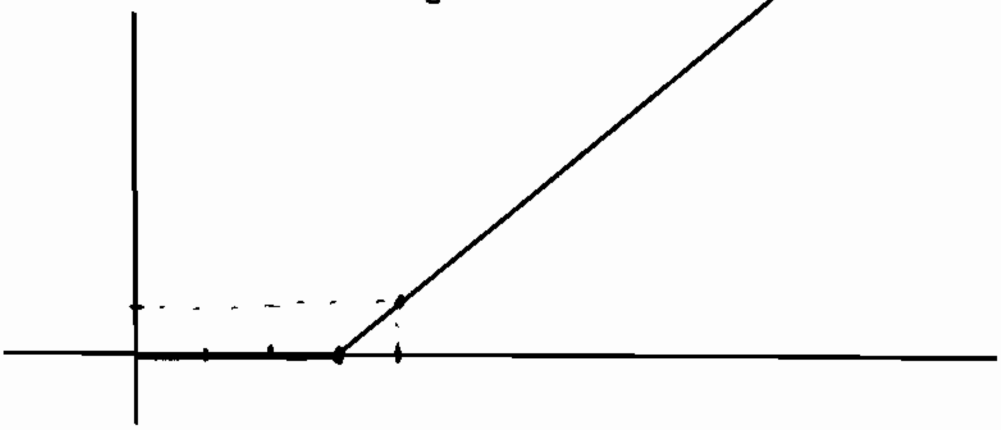
a) $f_1(t) = H(t-a) = \begin{cases} 0 & \text{si } t < a \\ 1 & \text{si } t \geq a \end{cases}$

Vamos a suponer que $a > 0$. Entonces podemos poner:

$$f_1(t) = \begin{cases} 0 & \text{si } 0 \leq t < a \\ 1 & \text{si } t \geq a \end{cases}$$

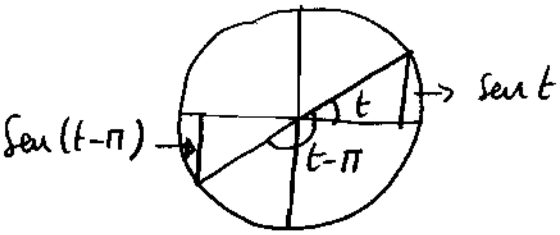


b) $f_2(t) = (t-3) \cdot H(t-3) = \begin{cases} 0 & \text{si } 0 \leq t < 3 \\ t-3 & \text{si } t \geq 3 \end{cases}$



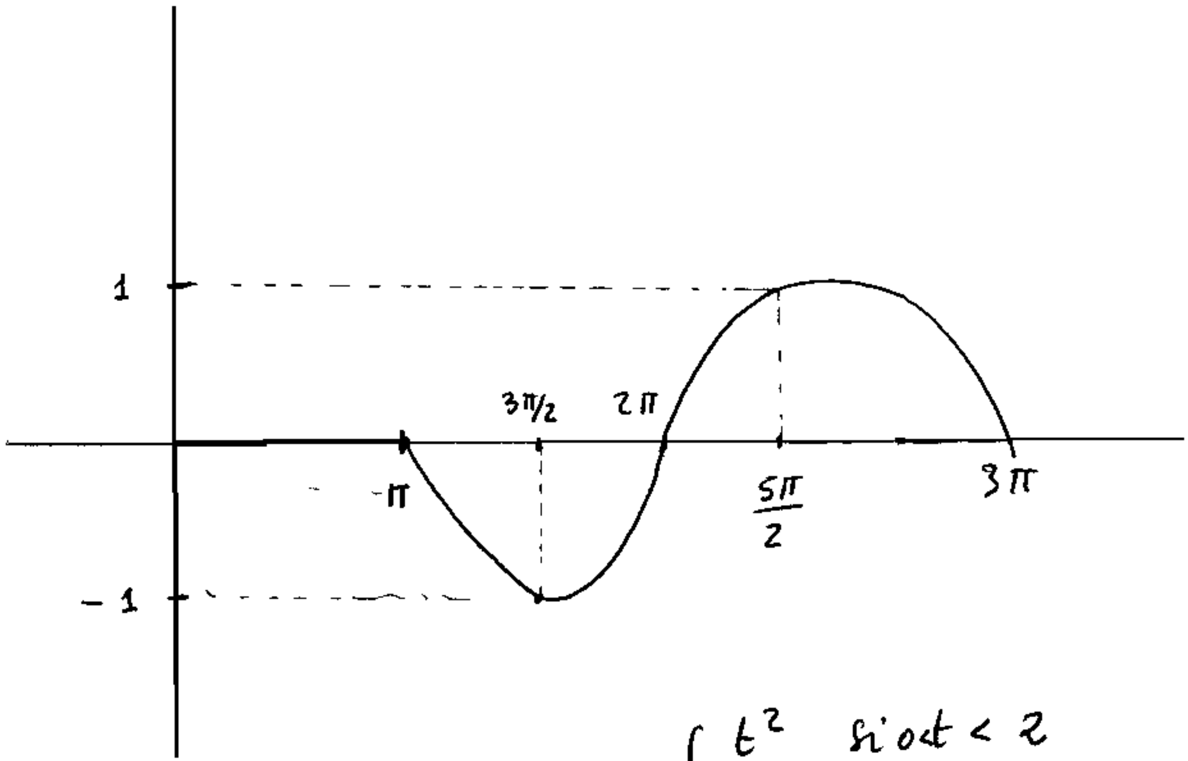
d) $f_4(t) = \operatorname{sen}(t-\pi) \cdot H(t-\pi)$

En primer lugar tenemos que $\operatorname{sen}(t-\pi) = -\operatorname{sen} t$.

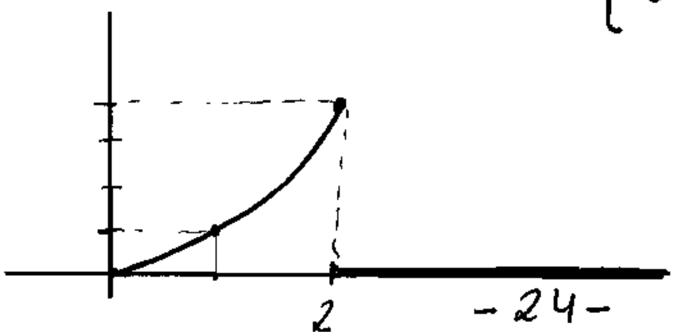


Por tanto podemos definir

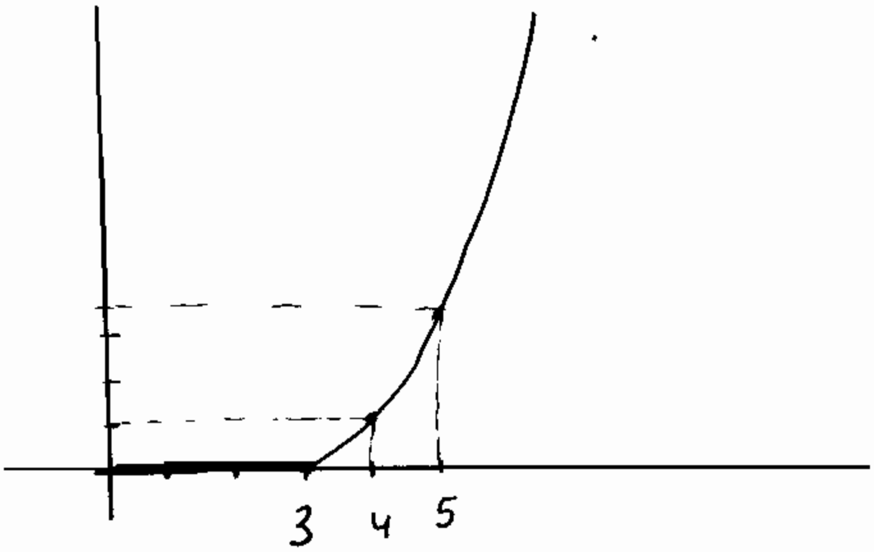
$$f_4(t) = \begin{cases} 0 & \text{si} \quad 0 < t < \pi \\ -\operatorname{sen} t & \text{si} \quad t \geq \pi \end{cases}$$



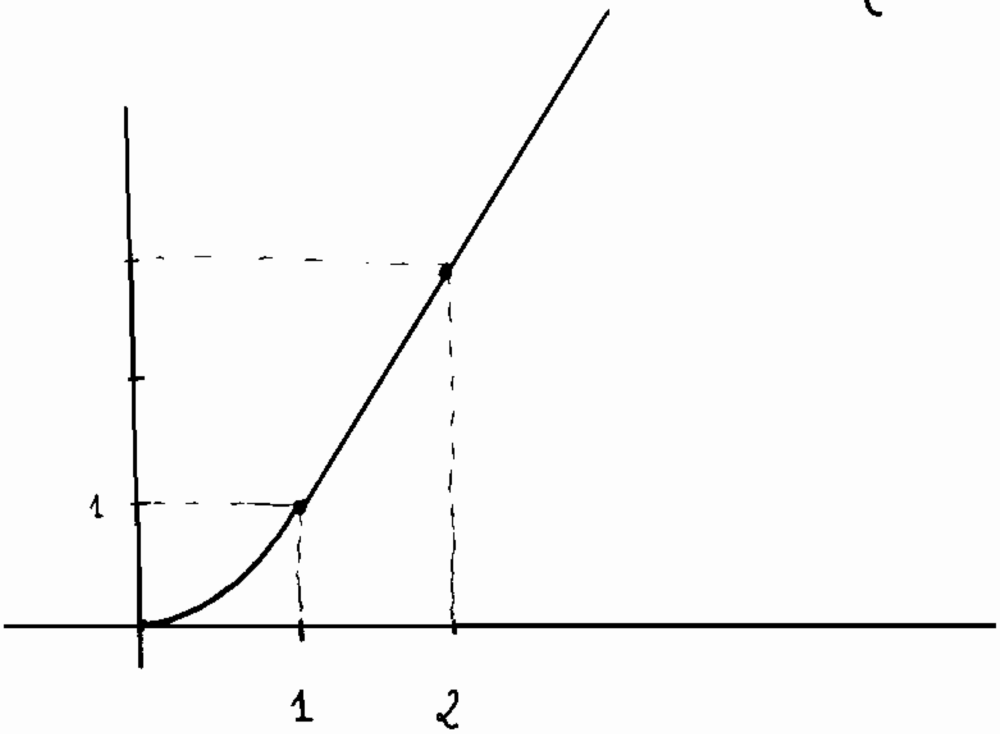
c) $f_3(t) = t^2 - t^2 H(t-2) = \begin{cases} t^2 & \text{si } 0 < t < 2 \\ 0 & \text{si } t \geq 2 \end{cases}$



e) $f_5(t) = (t-3)^2 \mathcal{U}(t-3) = \begin{cases} 0 & \text{si } 0 < t < 3 \\ (t-3)^2 & \text{si } t \geq 3 \end{cases}$



f) $f_6(t) = t^2 - (t-1)^2 \mathcal{U}(t-1) = \begin{cases} t^2 & \text{si } 0 \leq t < 1 \\ 2t-1 & \text{si } t \geq 1 \end{cases}$



10.- Determinar la transformada de Laplace de las siguientes funciones utilizando la función de Heaviside.

a) $f_1(t) = \begin{cases} 3 & \text{si } 0 \leq t < 1 \\ t & \text{si } t \geq 1 \end{cases}$

Podemos poner que $f_1(t) = 3 - (t-3) \cdot H(t-1)$. Por tanto:

$$\mathcal{L}[f_1(t)](z) = 3\mathcal{L}[1](z) - \mathcal{L}\left[(t-3)H(t-1)\right](z) =$$

$$= 3\mathcal{L}[1](z) - \mathcal{L}\left[\left(-2+(t-1)\right)H(t-1)\right](z) =$$

$$= 3\mathcal{L}[1](z) + 2\mathcal{L}[H(t-1)] - \mathcal{L}[(t-1)H(t-1)] =$$

$$= \frac{3}{z} + \frac{2e^{-z}}{z} - \frac{e^{-z}}{z^2}$$

Se tiene que:

$$\mathcal{L}[H(t-a)](z) = \frac{e^{-az}}{z}$$

$\mathcal{L}[f(t-a) \cdot H(t-a)](z) = e^{-az} \cdot \mathcal{L}[f(t)](z)$. Luego:

$$\mathcal{L}[H(t-1)] = \frac{e^{-z}}{z}$$

$$\mathcal{L}[(t-1) \cdot H(t-1)] = e^{-z} \cdot \mathcal{L}[t] = \frac{e^{-z}}{z^2}$$

$$\mathcal{L}[1] = \int_0^{+\infty} e^{-tz}\,dt = \lim_{b\to+\infty} \left.\frac{e^{-tz}}{-z}\right|_0^b = \lim_{b\to\infty} \frac{e^{-zb}}{-z} + \frac{1}{z} = \frac{1}{z}$$

b) $f_2(t) = \begin{cases} 4 & \text{si } 0 \le t < 2 \\ 2t-1 & \text{si } t \geq 2 \end{cases}$

Podemos poner que $f_2(t) = 4 + (2t-5)H(t-2)$

Por lo tanto:

$$\mathcal{L}[f_2(t)](z) = 4\mathcal{L}[4](z) + \mathcal{L}[(2t-5)H(t-2)] =$$

$$= 4\mathcal{L}[4](z) + \mathcal{L}[2(t-2)H(t-2) - H(t-2)](z) =$$

$$= 4\mathcal{L}[4](z) + 2\mathcal{L}[(t-2)H(t-2)] - \mathcal{L}[H(t-2)](z) =$$

$$= \frac{4}{z} + 2e^{-2z}\mathcal{L}[t](z) - \frac{e^{-2z}}{z}.$$ Como:

$$\mathcal{L}[t^n](z) = \frac{n!}{z^{n+1}} \Rightarrow \mathcal{L}[t](z) = \frac{1}{z^2}$$

Luego:

$$\mathcal{L}[f_2(t)](z) = \frac{4}{z} + 2 \cdot \frac{e^{-2z}}{z^2} - \frac{e^{-2z}}{z}$$

c) $f_3(t) = \begin{cases} t^2 & \text{si } 0 \le t < 2 \\ 3 & \text{si } t \geq 2 \end{cases}$

Se tiene que $f_3(t) = t^2 + (3-t^2)H(t-2)$

Podemos poner que

$$f_3(t) = t^2 + (3-t^2)\mathcal{H}(t-2) = t^2 + \left[(t-2)^2 + 4(t-2) + 1\right]\mathcal{H}(t-2)$$

Por lo tanto:

$$\mathcal{L}[f_3(t)](z) = \mathcal{L}[t^2](z) + \mathcal{L}\left[(t-2)^2\mathcal{H}(t-2)\right](z) +$$

$$+ 4\,\mathcal{L}[(t-2)\mathcal{H}(t-2)] + \mathcal{L}[\mathcal{H}(t-2)] \Rightarrow$$

$$\Rightarrow \mathcal{L}[f_3(t)](z) = \frac{4}{z^3} + \frac{2e^{-2z}}{z^3} + \frac{4e^{-2z}}{z^2} + \frac{e^{-2z}}{z}$$

- $\mathcal{L}[t^2](z) = \dfrac{2!}{z^3} = \dfrac{2}{z^3}$
- $\mathcal{L}\left[(t-2)^2\mathcal{H}(t-2)\right](z) = e^{-2z}\cdot\mathcal{L}[t^2](z) = \dfrac{2e^{-2z}}{z^3}$
- $\mathcal{L}[\mathcal{H}(t-2)](z) = \dfrac{e^{-2z}}{z}$
- $\mathcal{L}[(t-2)\mathcal{H}(t-2)] = e^{-2z}\cdot\mathcal{L}[t](z) = \dfrac{e^{-2z}}{z^2}$

d) $$f_4(t) = \begin{cases} e^{-t} & \text{si } 0 \le t < 2 \\ 0 & \text{si } t \geq 2 \end{cases}$$

$f_4(t) = e^{-t} - e^{-t}\mathcal{H}(t-2)$. Por tanto:

$$\mathcal{L}[f_4(t)](z) = \mathcal{L}[e^{-t}](z) - \mathcal{L}\left[e^{-t}\mathcal{H}(t-2)\right](z)$$

Es decir que:

$$\mathcal{L}[f_4(t)](z) = \mathcal{L}[e^{-t}](z) - \mathcal{L}[e^{-(t-2)} \cdot e^{-2} \cdot \mathcal{H}(t-2)](z) =$$

$$= \mathcal{L}[e^{-t}](z) - e^{-2} \cdot \mathcal{L}[e^{-(t-2)} \cdot \mathcal{H}(t-2)](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[f_4(t)] = \frac{1}{z+1}\left(1 + e^{-2t}\right)$$

- $\mathcal{L}[e^{-t}](z) = \dfrac{1}{z+1}$

- $\mathcal{L}[e^{-(t-2)} \cdot \mathcal{H}(t-2)] = e^{-2t} \cdot \mathcal{L}[e^{-t}] = \dfrac{e^{-2t}}{z+1}$

e) $$f_5(t) = \begin{cases} \operatorname{sen} 3t & \text{si } 0 \le t \le \frac{\pi}{2} \\ 0 & \text{si } t \geq \frac{\pi}{2} \end{cases}$$

$$f_5(t) = \operatorname{sen} 3t - \operatorname{sen} 3t \; \mathcal{H}\left(t - \frac{\pi}{2}\right)$$

Recordemos que $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$. Por tanto

$$\cos 3\left(t - \frac{\pi}{2}\right) = \cos\left(3t - \frac{3\pi}{2}\right) = \cos 3t \cdot \overbrace{\cancel{\cos\frac{3\pi}{2}}}^{=0} - \operatorname{sen} 3t \cdot \overbrace{\cancel{\operatorname{sen}\frac{3\pi}{2}}}^{=-1} \Rightarrow$$

$$\Rightarrow \cos 3\left(t - \frac{\pi}{2}\right) = + \operatorname{sen} 3t \text{ . Luego:}$$

$$f_5(t) = \operatorname{sen} 3t - \cos 3\left(t - \frac{\pi}{2}\right) \cdot \mathcal{H}\left(t - \frac{\pi}{2}\right).$$

$$\mathcal{L}[f_5(t)](z) = \mathcal{L}[\operatorname{sen} 3t](z) - \mathcal{L}\left[\cos 3\left(t-\frac{\pi}{2}\right) H\left(t-\frac{\pi}{2}\right)\right](z) =$$

$$= \frac{3}{z^2+9} - e^{-\frac{\pi}{2}z} \cdot \frac{z}{z^2+9}$$

- $$\mathcal{L}[\operatorname{sen} 3t](z) = \int_0^{+\infty} e^{-tz} \operatorname{sen} 3t \, dt = \lim_{b\to+\infty} \int_0^b e^{-tz} \operatorname{sen} 3t \, dt =$$

$$= \lim_{b\to+\infty} \frac{z \cdot e^{-bz}\left(\operatorname{sen} 3b - \frac{3}{z}\cos 3b\right)}{z^2+9} + \frac{3}{z^2+9} = \frac{3}{z^2+9}$$

$$I = \int e^{-tz} \operatorname{sen} 3t \, dt \text{ (por partes reiteradas) } \left\{ \begin{array}{l} u = \operatorname{sen} 3t;\ du = 3\cos 3t\, dt \\ dv = e^{-tz} dt \Rightarrow v = -\dfrac{e^{-tz}}{z} \end{array} \right\}$$

$$I = -\frac{1}{z} \cdot e^{-tz} \operatorname{sen} 3t + \frac{3}{z} \int e^{-tz} \cos 3t \, dt \quad \left\{ \begin{array}{l} u = \cos 3t \Rightarrow du = -3\operatorname{sen} 3t\, dt \\ dv = e^{-tz} dt \Rightarrow v = -\dfrac{e^{-tz}}{z} \end{array} \right\}$$

$$I = -\frac{1}{z} e^{-tz} \operatorname{sen} 3t + \frac{3}{z}\left[-\frac{1}{z} e^{-tz} \cos 3t - \frac{3}{z} I\right]$$

$$I\left(1 + \frac{9}{z^2}\right) = \frac{1}{z} \cdot e^{-tz}\left(\operatorname{sen} 3t - \frac{3}{z}\cos 3t\right)$$

$$I = \frac{z \cdot e^{-tz}\left(\operatorname{sen} 3t - \frac{3}{z}\cos 3t\right)}{z^2+9}$$

$$\int_0^b e^{-tz} \operatorname{sen} 3t \, dt = \frac{z \cdot e^{-bz}\left(\operatorname{sen} 3b - \frac{3}{z}\cos 3b\right)}{z^2+9} + \frac{3}{z^2+9}$$

- $$\mathcal{L}\left[\cos 3\left(t-\frac{\pi}{2}\right) H\left(t-\frac{\pi}{2}\right)\right](z) = e^{-\frac{\pi}{2}z} \cdot \mathcal{L}[\cos 3t](z) = e^{-\frac{\pi}{2}z} \cdot \frac{z}{z^2+9}$$

$$\mathcal{L}[\cos 3t](z) = \int_0^{+\infty} e^{-tz}\cos 3t\, dt = \lim_{b\to+\infty}\int_0^b \cos 3t\cdot e^{-tz}dt$$

$$= \lim_{b\to+\infty} \frac{-z\cdot e^{-bz}\left(\cos 3b - \frac{3}{z}\operatorname{sen} 3b\right)}{z^2+9} + \frac{z}{z^2+9} = \frac{z}{z^2+9}$$

$$I = \int \cos 3t\cdot e^{-tz}dt \text{ (por partes reiteradas) } \begin{cases} u = \cos 3t \Rightarrow du = -3\operatorname{sen} 3t\, dt \\ dv = e^{-tz}dt \Rightarrow v = -\frac{e^{-tz}}{z}\end{cases}$$

$$I = -\frac{1}{z}\cos 3t\cdot e^{-tz} - \frac{3}{z}\int e^{-tz}\operatorname{sen} 3t\, dt \quad \begin{cases} u = \operatorname{sen} 3t \Rightarrow du = 3\cos 3t\, dt \\ dv = e^{-tz}dt \Rightarrow v = -\frac{e^{-tz}}{z}\end{cases}$$

$$I = -\frac{1}{z}\cos 3t\, e^{-tz} - \frac{3}{z}\left(-\frac{1}{z}\operatorname{sen} 3t\, e^{-tz} + \frac{3}{z} I\right)$$

$$I\left(1+\frac{9}{z^2}\right) = -\frac{1}{z}e^{-tz}\left(\cos 3t - \frac{3}{z}\operatorname{sen} 3t\right)$$

$$I = \frac{-z\, e^{-tz}\left(\cos 3t - \frac{3}{z}\operatorname{sen} 3t\right)}{z^2+9}$$

$$\int_0^b \cos 3t\, e^{-tz}dt = \frac{-z\cdot e^{-bz}\left(\cos 3b - \frac{3}{z}\operatorname{sen} 3b\right)}{z^2+9} + \frac{z}{z^2+9}$$

f) $f_6(t)\begin{cases} \operatorname{sen} 3t & \text{si } 0\le t<\pi \\ 0 & \text{si } t\geq \pi\end{cases}$

Podemos poner $f_6(t) = \operatorname{sen} 3t - \operatorname{sen} 3t\cdot H(t-\pi)$

Teniendo en cuenta que $\operatorname{sen}(\alpha-\beta) = \operatorname{sen}\alpha\cos\beta - \cos\alpha\operatorname{sen}\beta$

se verifica:

$$\operatorname{sen} 3(t-\pi) = \operatorname{sen}(3t-3\pi) = \operatorname{sen} 3t \underbrace{\cos 3\pi}_{=-1} - \cos 3t \underbrace{\operatorname{sen} 3\pi}_{=0} = -\operatorname{sen} 3t$$

Luego:

$$f_6(t) = \operatorname{sen} 3t + \operatorname{sen} 3(t-\pi)\cdot H(t-\pi)$$

$$\mathcal{L}[f_6(t)](z) = \mathcal{L}[\operatorname{sen} 3t](z) + \mathcal{L}[\operatorname{sen} 3(t-\pi) H(t-\pi)](z) =$$

$$= \frac{3}{z^2+9} + e^{-\pi z}\cdot \mathcal{L}[\operatorname{sen} 3t](z) = \frac{3}{z^2+9} + e^{-\pi z}\cdot\frac{3}{z^2+9} \Rightarrow$$

$$\Rightarrow \mathcal{L}[f_6(t)](z) = \frac{3}{z^2+9}\left(1 + e^{-\pi z}\right)$$

g) $$f_7(t) = \begin{cases} t^2 & \text{si } 0 \le t < 1 \\ 3 & \text{si } 1 \le t < 2 \\ 0 & \text{si } t \geq 2 \end{cases}$$

Podemos expresar $f_7(t)$ del siguiente modo:

$$f_7(t) = t^2 - t^2 H(t-1) + 3[H(t-1) - H(t-2)] \Rightarrow$$

$$\Rightarrow f_7(t) = t^2 - \left[(t-1)^2 + 2(t-1) + 1\right] H(t-1) + 3H(t-1) - 3H(t-2)$$

$$\Rightarrow f_7(t) = t^2 - (t-1)^2 H(t-1) + 5(t-1)H(t-1) + H(t-1) - 3H(t-2)$$

Por lo tanto:

$$\mathcal{L}[f_7(t)](z) = \mathcal{L}[t^2](z) - \mathcal{L}[(t-1)^2 \cdot H(t-1)](z) +$$

$$+5\,\mathcal{L}[(t-1)H(t-1)](z) + \mathcal{L}[H(t-1)](z) - 3\,\mathcal{L}[H(t-2)](z) =$$

$$= \frac{2!}{z^3} - e^{-z} \cdot \mathcal{L}[t^2](z) + 5 \cdot e^{-z} \cdot \mathcal{L}[t](z) + \frac{e^{-z}}{z} - 3 \cdot \frac{e^{-2z}}{z}$$

$$\mathcal{L}[f_7(t)] = \frac{2}{z^3} - 2\,\frac{e^{-z}}{z^3} + 5 \cdot \frac{e^{-z}}{z^2} + \frac{e^{-z}}{z} - 3 \cdot \frac{e^{-2z}}{z}$$

h) $$f_8(t) = \begin{cases} t^2 & \text{si } 0 \le t < 2 \\ t-1 & \text{si } 2 \le t < 3 \\ 7 & \text{si } t \geq 3 \end{cases}$$

Podemos poner que:

$$f_8(t) = t^2 \cdot H(2-t) + (t-1) \cdot H(t-2) + (8-t) \cdot H(t-3)$$

Como se verifica la propiedad:

$\boxed{H(a-t) = 1 - H(t-a)}$ (esta propiedad es mía, la he inventado yo)

$$f_8(t) = t^2 - t^2 H(t-2) + (t-1)H(t-2) + (8-t)H(t-3) \Rightarrow$$

$$\Rightarrow f_8(t) = t^2 - [(t-2)^2 + 2(t-2)]H(t-2) + [(t-2)+1]H(t-2) +$$

$$+ (8-t)H(t-3)\text{, es decir:}$$

$$f_8(t) = t^2 + (t-2)^2 H(t-2) + 3(t-2)H(t-2) + H(t-2) - [(t-3)-5]H(t-3)$$

$$f_8(t)= t^2+(t-2)^2 H(t-2)+3(t-2)H(t-2)+H(t-2)-(t-3)H(t-3)+5H(t-3)$$

y por tanto:

$$\mathcal{L}[f_8(t)]= \mathcal{L}[t^2](z)+\mathcal{L}[(t-2)^2 H(t-2)](z)+3\mathcal{L}[(t-2)H(t-2)](z)+$$

$$+\mathcal{L}[H(t-2)](z)-\mathcal{L}[(t-3)H(t-3)]-5\mathcal{L}[H(t-3)](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[f_8(t)]= \frac{2}{z^3}+e^{-2z}\cdot\mathcal{L}[t^2](z)+3e^{-2z}\cdot\mathcal{L}[t](z)+\frac{e^{-2z}}{z}-$$

$$-e^{-3z}\cdot\mathcal{L}[t](z) \Rightarrow$$

$$\mathcal{L}[f_8(t)](z)= \frac{2}{z^3}+e^{-2z}\cdot\frac{2}{z^3}+3e^{-2z}\cdot\frac{1}{z^2}+\frac{e^{-2z}}{z}-e^{-3z}\cdot\frac{1}{z^2}$$

$$\mathcal{L}[f_8(t)](z)= \frac{2}{z^3}+e^{-2z}\left(\frac{2}{z^3}+\frac{3}{z^2}+\frac{1}{z}\right)-e^{-3z}\cdot\frac{1}{z^2}$$

11º Determinar

a) $\mathcal{L}^{-1}\left[\frac{5e^{-3z}}{z}-\frac{e^{-z}}{z}\right](t)$

Por la linealidad del inverso de Laplace:

$$\mathcal{L}^{-1}\left[\frac{5e^{-3z}}{z}-\frac{e^{-z}}{z}\right](t)= 5\cdot\mathcal{L}^{-1}\left[\frac{e^{-3z}}{z}\right](t)-\mathcal{L}^{-1}\left[\frac{e^{-z}}{z}\right](t)$$

Recordemos que:

i) $\mathcal{L}[H(t-a)](z)=\frac{e^{-az}}{z} \Leftrightarrow \mathcal{L}^{-1}\left[\frac{e^{-az}}{z}\right](t)= H(t-a)$

ii) $\mathcal{L}[f(t-a)H(t-a)](z)= e^{-az}\cdot\mathcal{L}[f(t)](z) \Leftrightarrow \mathcal{L}^{-1}\left[e^{-az}\cdot\mathcal{L}[f(t)](z)\right](t)=f(t-a)\cdot H(t-a)$

Por lo tanto:

$$\mathcal{L}^{-1}\left[\frac{5e^{-3z}}{z}-\frac{e^{-z}}{z}\right](t)= 5H(t-3)-H(t-1)$$

b) $\mathcal{L}^{-1}\left[\frac{e^{-4z}}{(z+2)^3}\right](t)$

Hacemos uso de la siguiente propiedad, que no se ha puesto en los apuntes pero que es muy importante:

$$\boxed{\mathcal{L}^{-1}\left[e^{-at}\cdot F(z)\right](t)=H(t-a)\cdot \mathcal{L}^{-1}\left[F(z)\right](t-a)}$$

$F(z)$ es una función cualquiera para la que existe transformada inversa.

Luego:

$$\mathcal{L}^{-1}\left[\frac{e^{-4z}}{(z+2)^3}\right](t)= H(t-4)\cdot \mathcal{L}^{-1}\left[\frac{1}{(z+2)^3}\right](t-4)=$$

$$=$$

$$\mathcal{L}^{-1}\left[\frac{1}{(z+2)^3}\right](t)= \text{Res}\left[\frac{e^{tz}}{(z+2)^3},-2\right]=\lim_{z\to -2}\frac{1}{2!}\frac{d^2}{dz^2}\left(e^{tz}\right)$$

$$=\lim_{z\to -2}\frac{1}{2}\cdot t^2e^{tz}=\frac{1}{2}t^2\cdot e^{-2t}$$

Por lo tanto:

$$\mathcal{L}^{-1}\left[\frac{1}{(z+2)^3}\right](t-4) = \frac{1}{2}(t-4)^2 \cdot e^{-2(t-4)}$$ y entonces:

$$\mathcal{L}^{-1}\left[\frac{e^{-4z}}{(z+2)^3}\right](t) = H(t-4)\cdot\frac{1}{2}(t-4)^2 e^{-2(t-4)}$$

c) $$\mathcal{L}^{-1}\left[\frac{e^{-3z}}{(z+1)^3}\right](t) = H(t-3)\cdot\mathcal{L}^{-1}\left[\frac{1}{(z+1)^3}\right](t-3)$$

$$\mathcal{L}^{-1}\left[\frac{1}{(z+1)^3}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{(z+1)^3}, -1\right] = \lim_{z\to -1}\frac{1}{2!}\frac{d^2}{dz^2}(e^{tz}) =$$

$$= \lim_{z\to -1}\frac{1}{2}\cdot t^2 e^{tz} = \frac{1}{2}t^2 e^{-t}.$$ Luego:

$$\mathcal{L}^{-1}\left[\frac{1}{(z+1)^3}\right](t-3) = \frac{1}{2}(t-3)^2 e^{-t+3}$$ y entonces:

$$\mathcal{L}^{-1}\left[\frac{e^{-3z}}{(z+1)^3}\right](t) = H(t-3)\cdot\frac{1}{2}\cdot(t-3)^2 \cdot e^{-t+3}$$

d) $$\mathcal{L}^{-1}\left[\frac{(1-e^{-2z})(1-3e^{-2z})}{z^2}\right](t) = \mathcal{L}^{-1}\left[\frac{1-4e^{-2z}+3e^{-4z}}{z^2}\right](t) =$$

$$= \mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) - 4\cdot\mathcal{L}^{-1}\left[\frac{e^{-2z}}{z^2}\right](t) + 3\cdot\mathcal{L}^{-1}\left[\frac{e^{-4z}}{z^2}\right](t)$$

Se tiene que:

- $\mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) = t$

- $\mathcal{L}^{-1}\left[\dfrac{e^{-2z}}{z^2}\right](t) = H(t-2)\cdot \mathcal{L}^{-1}\left[\dfrac{1}{z^2}\right](t-2)$

$\mathcal{L}^{-1}\left[\dfrac{1}{z^2}\right](t) = t \Rightarrow \mathcal{L}^{-1}\left[\dfrac{1}{z^2}\right](t-2) = t-2 \Rightarrow$

$\Rightarrow \mathcal{L}^{-1}\left[\dfrac{e^{-2z}}{z^2}\right](t) = H(t-2)\,(t-2)$

- $\mathcal{L}^{-1}\left[\dfrac{e^{-4z}}{z^2}\right](t) = H(t-4)\cdot \mathcal{L}^{-1}\left[\dfrac{1}{z^2}\right](t-4) =$

$= H(t-4)\cdot(t-4)$

luego:

$$\mathcal{L}^{-1}\left[\frac{(1-e^{-2z})(1-3e^{-2z})}{z^2}\right](t) = t - 4(t-2)H(t-2) + 3(t-4)H(t-4$$

12) Resolver los siguientes problemas de valor inicial usando la transformada de Laplace. Verificar la solución.

a) $\begin{cases} y'' + y = f_1(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$ donde $f_1(t) = \begin{cases} 4 & \text{si } 0 \le t < 2 \\ t+2 & \text{si } t \geq 2 \end{cases}$

Podemos poner que $f_1(t) = 4 + (t-2)\cdot H(t-2)$. Si aplicamos la transformada de Laplace a $y''+y = f_1(t)$ obtenemos:

$\mathcal{L}[y''](z) + \mathcal{L}[y'](z) = \mathcal{L}[f_1(t)](z)$

$z^2 \cdot \mathcal{L}[y](z) - z\,y(0) - y'(0) + \mathcal{L}[y](z) = 4\,\mathcal{L}[1] + \mathcal{L}[(t-2)H(t-2)](z)$

Poniendo $Y(z) = \mathcal{L}[y](z)$ queda:

$z^2 Y(z) + Y(z) = \frac{4}{z} + e^{-2z} \cdot \frac{1}{z^2}$, es decir:

$$Y(z)(z^2+1) = \frac{4z + e^{-2z}}{z^2} \Rightarrow Y(z) = \frac{4z + e^{-2z}}{z^2(z^2+1)} \Rightarrow$$

$\Rightarrow Y(z) = \frac{4z}{z^2(z^2+1)} + \frac{e^{-2z}}{z^2(z^2+1)}$. La solución del problema viene dada por:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{4\cancel{z}}{z^{\cancel{2}}(z^2+1)}\right](t) + \mathcal{L}^{-1}\left[\frac{e^{-2z}}{z^2(z^2+1)}\right](t)$$

- Vamos a calcular $\mathcal{L}^{-1}\left[\frac{4}{z(z^2+1)}\right](t)$. Para ello descomponemos en fracciones simples:

$$\frac{4}{z(z^2+1)} = \frac{A}{z} + \frac{Bz + C}{z^2+1} \Rightarrow Az^2 + A + Bz^2 + Cz = 4$$

$$\left.\begin{array}{l} A+B=0 \\ C=0 \\ A=4 \end{array}\right. \Rightarrow \frac{4}{z(z^2+1)} = \frac{4}{z} - \frac{4}{z^2+1}$$

$$\mathcal{L}^{-1}\left[\frac{4}{z(z+1)^2}\right](t) = 4\,\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - 4\,\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right](t)$$

$\mathcal{L}^{-1}\left[\frac{1}{z}\right] = 1$

$\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right] = \text{Res}\left[\frac{e^{tz}}{z^2+1}, i\right] + \text{Res}\left[\frac{e^{tz}}{z^2+1}, -i\right]$

$= \frac{e^{ti}}{2i} - \frac{e^{-ti}}{2i} = \frac{e^{ti}-e^{-ti}}{2i} = \text{sen}\, t$

$p(z) = e^{tz} \begin{cases} p(i) = e^{ti} \\ p(-i) = e^{-ti} \end{cases}$

$q(z) = z^2+1 \Rightarrow q'(z) = 2z \begin{cases} q'(i) = 2i \\ q'(-i) = -2i \end{cases}$

Luego:

$\mathcal{L}^{-1}\left[\frac{4}{z(z+1)^2}\right] = 4 \cdot \mathcal{L}^{-1}\left[\frac{1}{z(z+1)^2}\right] = 4 \cdot (1 - \text{sen}\, t)$

- Calculemos ahora $\mathcal{L}^{-1}\left[\frac{e^{-2z}}{z^2(z^2+1)}\right](t) =$

$= H(t-2) \cdot \mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-2)$. Para calcular

$\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-2)$ descomponemos en fracciones simples:

$\frac{1}{z^2(z^2+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{Cz+D}{z^2+1} = \frac{Az(z^2+1)+B(z^2+1)+z^2(Cz+D)}{z^2(z^2+1)}$

$Az^3 + Az + Bz^2 + B + Cz^3 + Dz^2 = 1$

$$\left.\begin{array}{l} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{array}\right\} \begin{array}{l} C=0 \\ D=-1 \end{array} \Rightarrow \frac{1}{z^2(z^2+1)} = \frac{1}{z^2} - \frac{1}{z^2+1}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{z^2}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right](t) =$$

$$= t - \operatorname{sen} t \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-2) = (t-2) - \operatorname{sen}(t-2)$$

La solución del problema viene dada por:

$$y(t) = 4(1-\operatorname{sen} t) + (t-2)H(t-2) - \operatorname{sen}(t-2)H(t-2)$$

b) $\begin{cases} y'' + y = f_2(t) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ donde $f_2(t) = \begin{cases} 3 & \text{si } 0 \le t < 4 \\ 2t-5 & \text{si } t \geq 4 \end{cases}$

Solución:

$f_2(t) = 3 + 2(t-4)H(t-4)$. Aplicamos el operador transformada de Laplace a $y'' + y = f_2(t)$

$$\mathcal{L}[y''](z) + \mathcal{L}[y](z) = \mathcal{L}[f_2(t)](z)$$

$$z^2 \mathcal{L}[y](z) - zy(0) - y'(0) + \mathcal{L}[y](z) = \mathcal{L}[3+2(t-4)H(t-4)](z)$$

$$z^2\, Y(z) - z + Y(z) = 3\,\mathcal{L}[1](z) + 2\,e^{-4t}\cdot \mathcal{L}[t](z)$$

$$(z^2+1)\,Y(z) - z = 3\cdot\frac{1}{z} + 2\cdot e^{-4z}\cdot\frac{1}{z^2}$$

$$(z^2+1)\,Y(z) = \frac{3z+2e^{-4t}}{z^2} + z$$

$$Y(z) = \frac{3z+2e^{-4z}+z^3}{z^2(z^2+1)}$$

$$Y(z) = \frac{3z+z^3}{z^2(z^2+1)} + 2e^{-4z}\cdot\frac{1}{z^2(z^2+1)}$$

La solución viene dada por:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{3+z^2}{z(z^2+1)}\right](t) + 2\mathcal{L}^{-1}\left[e^{-4z}\frac{1}{z^2(z^2+1)}\right](t)$$

- Calculamos $\mathcal{L}^{-1}\left[\frac{3+z^2}{z(z^2+1)}\right]$. Descomponemos en fracciones simples:

$$\frac{3+z^2}{z(z^2+1)} = \frac{A}{z} + \frac{Bz+C}{z^2+1} = \frac{Az^2+A+Bz^2+Cz}{z(z^2+1)}$$

$$\left.\begin{array}{l} A+B = 1 \\ C=0 \\ A=3 \end{array}\right\} \begin{array}{l} B=-2 \\ \Rightarrow \end{array} \frac{3+z^2}{z(z^2+1)} = \frac{3}{z} - \frac{2z}{z^2+1}$$

Luego:

$$\mathcal{L}^{-1}\left[\frac{3+z^2}{z(z^2+1)}\right](t)=3\,\mathcal{L}^{-1}\left[\frac{1}{z}\right](t)-2\,\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t)$$

$$=\boxed{3-2\cos t}$$

$$\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t)=\operatorname{Res}\left[\frac{e^{tz}\cdot z}{z^2+1},i\right]+\operatorname{Res}\left[\frac{e^{tz}z}{z^2+1},-i\right]=$$

$$=\frac{i\,e^{ti}}{2i}+\frac{-i\cdot e^{-ti}}{-2i}=\cos t$$

$$p(z)=e^{tz}\cdot z\quad\begin{cases}p(i)=i\,e^{ti}\\ p(-i)=-i\,e^{-ti}\end{cases}$$

$$q(z)=z^2+1\Rightarrow q'(z)=2z\Rightarrow\begin{cases}q'(i)=2i\\ q'(-i)=-2i\end{cases}$$

$$\mathcal{L}^{-1}\left[e^{-4z}\cdot\frac{1}{z^2\cdot(z^2+1)}\right](t)=H(t-4)\cdot\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-4)$$

Por el ejercicio 12 a) (página 32) sabemos que:

$$\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t)=t-\operatorname{sen}t\text{, por tanto:}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-4)=\boxed{(t-4)-\operatorname{sen}(t-4).}$$

La solución de la ecuación diferencial con valores iniciales es:

$$y(t)=3-2\cos t+2\cdot H(t-4)\left[(t-4)-\operatorname{sen}(t-4)\right]$$

Aunque es un verdadero coñazo vamos a comprobarlo:

$$y(t) = 3 - 2\cos t + 2H(t-4)\left[(t-4) - \operatorname{sen}(t-4)\right]$$

$$y(0) = 3 - 2\cos 0 + 0 \cdot = 3 - 2 = 1$$

$$y'(t) = 2\operatorname{sen} t + 2H'(t-4)\cdot\left[(t-4) - \operatorname{sen}(t-4)\right] + 2H(t-4)\left[1 - \cos(t-4)\right]$$

$$y'(0) = 2\operatorname{sen} 0 + 2H'(-4)\left[-4 - \operatorname{sen}(-4)\right] + 0 = 0 + 0 + 0 = 0$$

$$H'(t-4) = \begin{cases} 0 & \text{si } 0 < t < 4 \\ 0 & \text{si } t \geq 4 \end{cases} \Rightarrow H'(t-4) = 0.$$

Como $H'(t-4) = 0$:

$$y'(t) = 2\operatorname{sen} t + 2H(t-4)\left[1 - \cos(t-4)\right]$$

$$y''(t) = 2\cos t + 2H'(t-4)\left[1 - \cos(t-4)\right] + 2H(t-4)\cdot\operatorname{sen}(t-4)$$

$$y''(t) = 2\cos t + 2\operatorname{sen}(t-4)H(t-4)$$

$$y'' + y = \cancel{2\cos t} + 2\operatorname{sen}(t-4)H(t-4) + 3 \cancel{- 2\cos t} +$$
$$+ 2H(t-4)\left[(t-4) - \operatorname{sen}(t-4)\right] = 3 + 2(t-4)H(t-4) =$$

$= f_2(t)$, la solución es pues correcta.

Advertencia: que me arranquen todas las muelas antes de que yo vuelva a comprobar otro bodrio como este, si quieres hazlo tú.

c) $$\begin{cases} y'' + y = f_3(t) \\ y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \text{donde } f_3(t) = \begin{cases} 1 & \text{si } 0 \leq t \leq \frac{\pi}{2} \\ 0 & \text{si } t \geq \frac{\pi}{2} \end{cases}$$

Se tiene que $f_3(t) = 1 - H(t - \frac{\pi}{2})$

Aplicando la transformada de Laplace a la ecuación

$$y'' + y = f_3(t) \Rightarrow \mathcal{L}[y''](z) + \mathcal{L}[y](z) = \mathcal{L}[f_3(t)](z)$$

$$z^2 \mathcal{L}[y](z) - z\,y(0) - y'(0) + \mathcal{L}[y](z) = \mathcal{L}[1](z) - \mathcal{L}[H(t-\tfrac{\pi}{2})](z)$$

$$z^2 \mathcal{L}[y](z) - 1 + \mathcal{L}[y](z) = \frac{1}{z} - \frac{e^{-\frac{\pi}{2}z}}{z}$$

Si hacemos $Y(z) = \mathcal{L}[y](z)$

$$z^2 Y(z) - 1 + Y(z) = \frac{1 - e^{-\frac{\pi}{2}z}}{z}$$

$$Y(z)\left[z^2 + 1\right] = \frac{1 - e^{-\frac{\pi}{2}z}}{z} + 1 \Rightarrow Y(z) = \frac{1 + z - e^{-\frac{\pi}{2}z}}{z(z^2+1)}$$

La solución viene dada por la transformada inversa:

$$y(t) = \mathcal{L}^{-1}\left[Y(z)\right](t) = \mathcal{L}^{-1}\left[\frac{1+z-e^{-\frac{\pi}{2}z}}{z(z^2+1)}\right](t), \text{ es decir:}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1+z}{z(z^2+1)}\right](t) - \mathcal{L}^{-1}\left[e^{-\frac{\pi}{2}z} \cdot \frac{1}{z(z^2+1)}\right](t)$$

• Calculamos $\mathcal{L}^{-1}\left[\dfrac{z+1}{z(z^2+1)}\right](t)$ y para ello descomponemos en fracciones simples:

$$\frac{z+1}{z(z^2+1)} = \frac{A}{z} + \frac{Bz+C}{z^2+1} \Rightarrow Az^2+A+Bz^2+Cz = z+1 \Rightarrow$$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow B=-1 \\ C=1 \\ A=1 \end{cases} \Rightarrow \frac{z+1}{z(z^2+1)} = \frac{1}{z} - \frac{z-1}{z^2+1}$$

$$\mathcal{L}^{-1}\left[\frac{z+1}{z(z^2+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - \mathcal{L}^{-1}\left[\frac{z-1}{z^2+1}\right](t) =$$

$$= \boxed{1 - \cos t + \operatorname{sen} t}$$

$$\mathcal{L}^{-1}\left[\frac{z-1}{z^2+1}\right](t) = \operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, i\right] + \operatorname{Res}\left[\frac{e^{tz}(z-1)}{z^2+1}, -i\right]$$

$$= \frac{p(i)}{q'(+i)} + \frac{p(-i)}{q'(-i)} = \frac{i\,e^{ti} + i \cdot e^{-ti}}{2i} - \frac{e^{ti} - e^{-ti}}{2i} = \boxed{\cos t - \operatorname{sen} t}$$

$$p(z) = e^{tz}(z-1) \Rightarrow p(i) = e^{ti}(i-1) \; ; \; p(-i) = e^{-ti}(-i-1)$$

$$q(z) = z^2+1 \Rightarrow q'(z) = 2z \Rightarrow q'(i) = 2i \; ; \; q'(-i) = -2i$$

• Calculamos $\mathcal{L}^{-1}\left[e^{-\frac{\pi}{2}z} \cdot \dfrac{1}{z(z^2+1)}\right] = u(t-\frac{\pi}{2}) \cdot \mathcal{L}^{-1}\left[\dfrac{1}{z(z^2+1)}\right](t-\frac{\pi}{2})$

$$= u(t-\tfrac{\pi}{2}) \cdot \left(1 - \cos\left(t - \tfrac{\pi}{2}\right)\right)$$

Para calcular $\mathcal{L}^{-1}\left[\dfrac{1}{z(z^2+1)}\right](t-\frac{\pi}{2})$ descomponemos en fracciones simples:

$$\frac{1}{z(z^2+1)} = \frac{A}{z} + \frac{Bz+D}{z^2+1} \Rightarrow Az^2+A+Bz^2+Dz = 1$$

$$\begin{aligned} &A+B=0 \\ &D=0 \quad \Rightarrow B=-1 \quad \Rightarrow \frac{1}{z(z^2+1)} = \frac{1}{z} - \frac{z}{z^2+1} \Rightarrow \\ &A=1 \end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z(z^2+1)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - \mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t) =$$

$$= \boxed{1 - \cos t}$$

$$\mathcal{L}^{-1}\left[\frac{z}{z^2+1}\right](t) = \operatorname{Res}\left[\frac{z\cdot e^{tz}}{z^2+1}, i\right] + \operatorname{Res}\left[\frac{z\cdot e^{tz}}{z^2+1}, -i\right]$$

$$= \frac{p(i)}{q'(i)} + \frac{p(-i)}{q'(-i)} = \frac{e^{ti}}{2} + \frac{e^{-ti}}{2} = \cos t$$

$$p(z) = z\cdot e^{tz} \Rightarrow \begin{cases} p(i) = i\, e^{ti} \\ p(-i) = -i\cdot e^{-ti} \end{cases}$$

$$q(z) = z^2+1 \Rightarrow q'(z) = 2z \begin{cases} q'(i) = 2i \\ q'(-i) = -2i \end{cases}$$

luego $\mathcal{L}^{-1}\left[\frac{1}{z(z^2+1)}\right]\left(t-\frac{\pi}{2}\right) = \boxed{1-\cos\left(t-\frac{\pi}{2}\right)}$

la solución del problema es:

$$y(t) = 1-\cos t + \operatorname{sen} t - H\left(t-\frac{\pi}{2}\right)\left[1-\cos\left(t-\frac{\pi}{2}\right)\right].$$

d) $\begin{cases} y''+4y = f_4(t) \\ y(0)=0 \\ y'(0) = 0 \end{cases}$ donde $f_4(t) = \text{sen}\, t - \mathcal{H}(t-2\pi)\,\text{sen}(t-2\pi)$

$$\mathcal{L}[y''](z)+4\,\mathcal{L}[y](z) = \mathcal{L}[\text{sen}\, t](z) - e^{-2\pi z}\cdot \mathcal{L}[\overset{\text{sen}\, t}{\overset{\shortparallel}{\text{sen}(t-2\pi)}}](z).$$

$$z^2\mathcal{L}[y](z) - zy(0) - y'(0) + \mathcal{L}[y](z) = \frac{1}{z^2+1} - e^{-2\pi z}\cdot \frac{1}{z^2+1}$$

Haciendo el cambio $\mathcal{L}[y](z) = Y(z)$ queda:

$$z^2\,Y(z) + Y(z) = \frac{1}{z^2+1}\left(1 - e^{-2\pi z}\right)$$

$$Y(z) = \frac{1}{(z^2+1)^2} - e^{-2\pi z}\cdot\frac{1}{(z^2+1)^2}$$

Por tanto la solucion es:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t) - \mathcal{L}^{-1}\left[e^{-2\pi z}\cdot\frac{1}{(z^2+1)^2}\right](t)$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t) - \mathcal{H}(t-2\pi)\cdot\mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t-2\pi)$$

Calculemos:

$$\mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t) = \text{Res}\left[\frac{e^{tz}}{(z^2+1)^2}, i\right] + \text{Res}\left[\frac{e^{tz}}{(z^2+1)^2}, -i\right]$$ y

hemos de tener en cuenta que tanto $z_0 = i$ como $z_1 = -i$

son polos dobles de $\dfrac{e^{tz}}{(z^2+1)^2}$.

$$\operatorname{Res}\left[\frac{e^{tz}}{(z^2+1)^2}, i\right] = \lim_{z\to i} \frac{d}{dz}\left[\cancel{(z-i)^2}\cdot \frac{e^{tz}}{\cancel{(z-i)^2}(z+i)^2}\right] =$$

$$= \lim_{z\to i} \frac{t\cdot e^{tz}(z+i) - 2e^{tz}}{(z+i)^3} = \frac{2ti\cdot e^{ti} - 2e^{ti}}{8i^3} =$$

$$= \frac{2ti\cdot e^{ti} - 2e^{ti}}{-8i} = \frac{tie^{ti} - e^{ti}}{-4i} = e^{ti}\,\frac{1-ti}{4i}$$

$$\operatorname{Res}\left[\frac{e^{tz}}{(z^2+1)^2}, -i\right] = \lim_{z\to -i} \frac{d}{dz}\left[\cancel{(z+i)^2}\cdot \frac{e^{tz}}{(z-i)^2\cancel{(z+i)^2}}\right] =$$

$$= \lim_{z\to -i} \frac{t\cdot e^{tz}(z-i)^2 - 2e^{tz}(z-i)}{(z-i)^4} = \lim_{z\to -i} \frac{te^{tz}(z-i) - 2e^{tz}}{(z-i)^3} =$$

$$= \frac{-2ti\,e^{-ti} - 2e^{-ti}}{-8i^3} = -\frac{ti\cdot e^{-ti} + e^{-ti}}{4i} =$$

$$= -e^{-ti}\,\frac{1+ti}{4i}$$

Luego:

$$\mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t) = \frac{1}{4i}\left[e^{ti} - tie^{ti} - e^{-ti} + ti\cdot e^{-ti}\right] =$$

$$= \frac{1}{2}\left[\frac{e^{ti} - e^{-ti}}{2i} - ti\,\frac{e^{ti} - e^{-ti}}{2i}\right] = \frac{1}{2}\left[\operatorname{sen} t - ti \operatorname{sen} t\right]$$

$$= \frac{1}{2}\operatorname{sen} t\,(1-ti)$$

Por tanto:

$$\mathcal{L}^{-1}\left[\frac{1}{(z^2+1)^2}\right](t-2\pi) = \frac{1}{2}\left[1-(t-2\pi)i\right]\cdot \operatorname{sen}(t-2\pi)$$

La solucion es entonces:

$$y(t) = \frac{1}{2}(1-t)\operatorname{sen} t - H(t-2\pi)\cdot\frac{1}{2}\left[1-(t-2\pi)i\right]\operatorname{sen}(t-2\pi)$$

13) Determinar $y(\frac{1}{2}\pi)$ e $y(2+\frac{1}{2}\pi)$ para la función $y(t)$ que satisface el problema de valor inicial:

$$\begin{cases} y'' + y = (t-2)\cdot H(t-2) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Veamos cual es la solución:

$$\mathcal{L}[y''](z) + \mathcal{L}[y](z) = \mathcal{L}[(t-2)H(t-2)](z)$$

$$z^2\mathcal{L}[y](z) - zy(0) - y'(0) + \mathcal{L}[y](z) = e^{-2z}\cdot\mathcal{L}[t](z)$$

$Y(z) = \mathcal{L}[y](z)$:

$$z^2Y(z) + Y(z) = e^{-2z}\cdot\frac{1}{z^2} \Rightarrow Y(z) = e^{-2z}\cdot\frac{1}{z^2(z^2+1)}$$

La solución viene dada por:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[e^{-2z} \cdot \frac{1}{z^2(z^2+1)}\right] \Rightarrow$$

$$y(t) = H(t-2) \cdot \mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-2)$$

Según el ejercicio 12 a) (página 32) sabemos que:

$\mathcal{L}^{-1}\left[\dfrac{1}{z^2(z^2+1)}\right](t) = t - \operatorname{sen} t$. Por tanto:

$$\mathcal{L}^{-1}\left[\frac{1}{z^2(z^2+1)}\right](t-2) = (t-2) - \operatorname{sen}(t-2)$$

La solución es pues:

$y(t) = H(t-2) \cdot [(t-2) - \operatorname{sen}(t-2)]$. Por tanto:

$$y\left(\tfrac{1}{2}\pi\right) = H\left(\tfrac{1}{2}\pi - 2\right)\left[\left(\tfrac{1}{2}\pi - 2\right) - \operatorname{sen}\left(\tfrac{1}{2}\pi - 2\right)\right] = 0$$

ya que $H\left(\frac{1}{2}\pi - 2\right) = 0$ al ser $\frac{1}{2}\pi < 2$

$$y\left(2 + \tfrac{1}{2}\pi\right) = H\left(\tfrac{1}{2}\pi\right)\left[\tfrac{1}{2}\pi - \operatorname{sen}\tfrac{\pi}{2}\right] = 1 \cdot \left(\tfrac{1}{2}\pi - 1\right) = \tfrac{1}{2}\pi - 1$$

14) Determinar $y(1)$ e $y(4)$ para la función $y(t)$ que satisface el problema de valor inicial:

$$\begin{cases} y'' + 2y' + y = 2 + (t-3)\,H(t-3) \\ y(0) = 2 \\ y'(0) = 1 \end{cases}$$

Solución:

$$\mathcal{L}[y''](z) + 2\,\mathcal{L}[y'](z) + \mathcal{L}[y](z) = 2\,\mathcal{L}[1](z) + \mathcal{L}[(t-3)H(t-3)](z)$$

$$z^2\,\mathcal{L}[y](z) - z\,y(0) - y'(0) + 2\left[z\,\mathcal{L}[y](z) - y(0)\right] + \mathcal{L}[y](z) =$$

$$= \frac{2}{z} + e^{-3z}\cdot \mathcal{L}[t](z)$$

Si hacemos $Y(z) = \mathcal{L}[y](z)$:

$$z^2 Y(z) - 2z - 1 + 2z\,Y(z) - 4 + Y(z) = \frac{2}{z} + e^{-3z}\cdot\frac{1}{z^2}$$

$$Y(z)\left[z^2 + 2z + 1\right] = \frac{2 + e^{-3z}}{z} + 5 + 2z$$

$$Y(z)\,(z+1)^2 = \frac{2z^2 + 5z + 2 + e^{-3z}}{z}$$

$$Y(z) = \frac{2z^2+5z+2+e^{-3z}}{z\,(z+1)^2} \Rightarrow Y(z) = \frac{2z^2+5z+2}{z(z+1)^2} + e^{-3z}\cdot\frac{1}{z\,(z+1)^2}$$

La solución viene dada por:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{2z^2+5z+2}{z(z+1)^2}\right](t) + \mathcal{L}^{-1}\left[e^{-3z}\cdot\frac{1}{z(z+1)^2}\right](t)$$

• Calculamos $\mathcal{L}^{-1}\left[\frac{2z^2+5z+2}{z\,(z+1)^2}\right](t)$, para ello descomponemos en fracciones simples:

$$\frac{2z^2+5z+2}{z(z+1)^2} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$, es decir:

$$A(z+1)^2 + Bz(z+1) + Cz = 2z^2+5z+2$$

$$Az^2+2Az+A+Bz^2+Bz+Cz = 2z^2+5z+2$$

$A+B=2 \Rightarrow B=0$ 2)

$2A+B+C=5 \Rightarrow C=1.$

$A=2$

Luego:

$$\frac{2z^2+5z+2}{z(z+1)^2} = \frac{2}{z} + \frac{1}{(z+1)^2} \Rightarrow \mathcal{L}^{-1}\left[\frac{2z^2+5z+2}{z(z+1)^2}\right](t) =$$

$$= 2\,\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t) = 2 + t\cdot e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{(z+1)^2}, -1\right] =$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2\cdot \frac{e^{tz}}{(z+1)^2}\right] = \lim_{z\to -1} t\cdot e^{tz} = t\cdot e^{-t}$$

Luego $\mathcal{L}^{-1}\left[\dfrac{2z^2+5z+2}{z(z+1)^2}\right](t) = 2+t\cdot e^{-t}$

• Calculamos ahora $\mathcal{L}^{-1}\left[e^{-3z}\cdot \dfrac{1}{z(z+1)^2}\right](t) =$

$$= H(t-3)\cdot \mathcal{L}^{-1}\left[\frac{1}{z(z+1)^2}\right](t-3)$$

$$\frac{1}{z(z+1)^2} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{(z+1)^2} \Rightarrow \begin{cases} A+B=0 \Rightarrow B=-1 \\ 2A+B+C=0 \Rightarrow C=-1 \\ A=1 \end{cases}$$

Luego:

$$\frac{1}{z(z+1)^2} = \frac{1}{z} - \frac{1}{z+1} - \frac{1}{(z+1)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z(z+1)^2}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{z}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{z+1}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t)$$

$$= 1 - e^{-t} - t \cdot e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z+1}\right] = \operatorname{Res}\left[\frac{e^{tz}}{z+1}, -1\right] = e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(z+1)^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{(z+1)^2}, -1\right] = \lim_{z\to -1} \frac{d}{dz}\left(e^{tz}\right) =$$

$$= \lim_{z\to -1} \left(t \cdot e^{tz}\right) = t \cdot e^{-t}$$

Luego:

$$\mathcal{L}^{-1}\left[\frac{1}{z(z+1)^2}\right](t-3) = 1 - e^{-t+3} - (t-3)\, e^{-t+3} \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left[e^{-3t} \cdot \frac{1}{z(z+1)^2}\right](t) = H(t-3)\left[1 - e^{-t+3}(t-2)\right]$$

La solución del problema es:

$$y(t) = 2 + t \cdot e^{-t} + H(t-3)\left[1 - e^{-t+3}(t-2)\right]$$

$$y(1) = 2 + \frac{1}{e}$$

$$y(4) = 2 + 4e^{-4} + 1 - \frac{2}{e}$$

15º Determinar las siguientes transformadas de Laplace mediante el teorema de convolución:

a) $F_1(z) = \dfrac{1}{z\,(z^2+K^2)}$

Solución:

Se define la convolución de f y g como:

$$(f*g)(t) = \begin{cases} 0 & \text{si } t<0 \\ \int_0^t f(s)\cdot g(t-s)\,ds = \int_0^t f(t-s)\cdot g(s)\,d \end{cases}$$

Se verifica que:

$$\mathcal{L}[(f*g)(t)](z) = \mathcal{L}[f(t)](z)\cdot\mathcal{L}[g(t)](z)$$

Lo que nos pide el ejercicio es hallar una función $f_1(t)$ tal que $\mathcal{L}[f_1(t)](z) = F_1(z)$, es decir que ~~Si se~~ verifica $F_1(z) = G_1(z)\cdot G_2(z)$ donde

$G_1(z) = \mathcal{L}[g(t)](z)$, $G_2(z) = \mathcal{L}[h(t)](z)$

Entonces $f_1(t) = g(t) * h(t) = \int_0^t g(t-s)\cdot h(s)\,ds$.

$$F_1(z) = \frac{1}{z}\cdot\frac{1}{z^2+K^2}$$

Luego:

$$F_2(z) = \mathcal{L}[4t] \cdot \mathcal{L}[e^{2t}](z) = \mathcal{L}[4t * e^{2t}](z)$$

La función $f_2(t) = 4t * e^{2t}$ verifica la propiedad:

$\mathcal{L}[f_2(t)](z) = F_2(z)$. Calculemos $f_2(t)$

$$f_2(t) = 4t * e^{2t} = \int_0^t 4(t-s)\cdot e^{2s}\,ds = \left(2t - 2s - \frac{1}{4}\right)e^{2s}\Big|_0^t =$$

$$= \left(2t - 2t - \frac{1}{4}\right)e^{2t} - \left(2t - 0 - \frac{1}{4}\right)e^{0} = -2t + \frac{1}{4} - \frac{1}{4}e^{2t}$$

$$\int 4(t-s)\cdot e^{2s}\,ds = 4\int t\cdot e^{2s}\,ds - 4\int s\,e^{2s}\,ds =$$

$$= 4t\cdot\frac{e^{2s}}{2} - 4\int s\,e^{2s}\,ds \text{ (por partes) } \begin{cases} u = s \Rightarrow du = ds \\ dv = e^{2s}ds \Rightarrow v = \dfrac{e^{2s}}{2} \end{cases}$$

$$= 4t\cdot\frac{e^{2s}}{2} - 4\cdot\left(\frac{s\cdot e^{2s}}{2} - \frac{1}{2}\int e^{2s}\,ds\right) =$$

$$= 2t\,e^{2s} - 2s\,e^{2s} - \frac{1}{4}e^{2s}$$

La solución a $\mathcal{L}[f_2(t)](z) = F_2(z) = \dfrac{4}{z^2(z-2)}$ es

$$f_2(t) = \frac{1}{4} - 2t - \frac{1}{4}e^{2t}$$

c) $F_3(z) = \dfrac{1}{z\cdot(z+2)}$

$F_3(z) = \dfrac{1}{z} \cdot \dfrac{1}{z+2}$

$\mathcal{L}^{-1}\left[\frac{1}{z}\right](t) = 1$

$\mathcal{L}^{-1}\left[\frac{1}{z+2}\right](t) = \text{Res}\left[\frac{e^{tz}}{z+2}, -2\right] = e^{-2t}$

Luego:

$F_3(z) = \mathcal{L}[1](z) \cdot \mathcal{L}[e^{-2t}](z) = \mathcal{L}[1 * e^{-2t}](z)$

de este modo la función $f_3 : [0, +\infty[\longrightarrow \mathbb{C}$ que verifica $\mathcal{L}[f_3(t)](z) = F_3(z)$ vale:

$$f_3(t) = 1 * e^{-2t} = \int_0^t 1 \cdot e^{-2s}\,ds = \frac{e^{-2s}}{-2}\Big|_0^t =$$

$$\frac{1}{2} - \frac{1}{2}e^{-2t} = \frac{1}{2}\left(1 - e^{-2t}\right) = f_3(t)$$

$$\mathcal{L}\left[\frac{1}{2}\left(1 - e^{-2t}\right)\right] = \frac{1}{z(z+2)}$$

d) $F_4(z) = \dfrac{1}{(z^2+1)^2}$

se tiene que $F_4(z) = \dfrac{1}{z^2+1} \cdot \dfrac{1}{z^2+1}$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+1}\right](t) = \text{Res}\left[\frac{e^{tz}}{z^2+1}, i\right] + \text{Res}\left[\frac{e^{tz}}{z^2+1}, -i\right] =$$

$$= \frac{e^{ti}}{2i} - \frac{e^{-ti}}{2i} = \text{Sen}\, t$$

Luego $F_4(z) = \mathcal{L}[\text{Sen}\, t] \cdot \mathcal{L}[\text{Sen}\, t] = \mathcal{L}[\text{Sen}\, t * \text{Sen}\, t]$

Entonces la función $f_4 : [0, +\infty[\longrightarrow \mathbb{C}$ que verifica

$\mathcal{L}[f_4(t)](z) = F_4(z)$ vale $f_4(t) = \text{Sen}\, t * \text{Sen}\, t$, es decir:

$$f_4(t) = \int_0^t \text{Sen}(t-s) \cdot \text{Sen}\, s\, ds$$

Teniendo en cuenta la fórmula de Mendoza (transformación de sumas en productos de funciones trigonométricas)

$$\text{Sen}\, A \cdot \text{Sen}\, B = -\frac{1}{2}\left[\text{Cos}(A+B) - \text{Cos}(A-B)\right]$$

$$\text{Sen}\, s \cdot \text{Sen}(t-s) = -\frac{1}{2}\left[\text{Cos}\, t - \text{Cos}(2s-t)\right]$$

tenemos que:

$$f_4(t) = \int_0^t -\frac{1}{2}\left[\text{Cos}\, t - \text{Cos}(2s-t)\right] dt = -\frac{1}{2}\left[s\,\text{Cos}\, t - \frac{1}{2}\,\text{Sen}(2s-t)\right]_0^t$$

$$f_4(t) = -\frac{1}{2}\left[t\,\text{Cos}\, t - \frac{1}{2}\,\text{Sen}\, t + \frac{1}{2}\,\text{Sen}(-t)\right] \Rightarrow f_4(t) = -\frac{1}{2}(t\,\text{Cos}\, t - \text{Sen}\, t)$$

16) Resolver los siguientes problemas de valor inicial utilizando el teorema de convolución:

$$y'' + 2y' + y = f_1(t)\,; \quad y(0) = 0\,; \quad y'(0) = 0.$$

Solución:

Me imagino que f_1 será la función del ejercicio 15a), es decir $f_1(t) = -\frac{1}{k}\cos t$ $\quad \mathcal{L}[f_1(t)] = F_1(z) = \dfrac{1}{z(z^2+k^2)}$

Aplicando la transformada de Laplace a la ecuación diferencial:

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[f_1(t)]$$

$$z^2\mathcal{L}[y](z) - z\,y(0) - y'(0) + 2\left[z\,\mathcal{L}[y](z) - y(0)\right] + \mathcal{L}[y](z) = F_1(z)$$

Si hacemos $\mathcal{L}[y](z) = Y(z)$

$$z^2\,Y(z) + 2z\,Y(z) + Y(z) = \frac{1}{z(z^2+k^2)}$$

$$Y(z)\cdot(z+1)^2 = \frac{1}{z(z^2+k^2)} \Rightarrow Y(z) = \frac{1}{z(z+1)^2(z^2+k^2)}$$

La solución del problema viene dada por:

$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\dfrac{1}{z(z^2+k^2)(z+1)^2}\right]$, es decir que tendremos que encontrar una función $g(t)$ que

que verifique:

$$\mathcal{L}[y(t)](z) = \frac{1}{z(z^2+k^2)(z+1)^2} = \frac{1}{z(z^2+k^2)} \cdot \frac{1}{(z+1)^2}$$

Como $\mathcal{L}[f_1(t)](z) = \mathcal{L}[\frac{1}{z(z^2+k^2)}]$

$$\mathcal{L}[t \cdot e^{-t}](z) = \frac{1}{(z+1)^2}$$

Tenemos que:

$$\mathcal{L}[y(t)](z) = \mathcal{L}[f_1(t)](z) \cdot \mathcal{L}[t \cdot e^{-t}](z) \Rightarrow$$

$$\Rightarrow \mathcal{L}[y(t)](z) = \mathcal{L}[f_1(t) * t \cdot e^{-t}](z) \Rightarrow$$

$$\Rightarrow y(t) = f_1(t) * t \cdot e^{-t} \Rightarrow y(t) = -\frac{1}{k}\cos t * t \cdot e^{-t}$$

La solución del problema de valores iniciales es:

$y(t) = -\frac{1}{k}\int_0^t \cos(t-s) \cdot s\, e^{-s}\, ds$. (La integral es larga y engorrosa, se hace por partes reiteradas)

b) $\begin{cases} y'' - k^2 y = f_2(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$ $\qquad f_2(t) = \frac{1}{4} - 2t - \frac{1}{4}e^{2t}$

$$\mathcal{L}[f_2(t)](z) = F_2(z) = \frac{4}{z^2(z-2)}$$

Solución:

$$\mathcal{L}[y''(t)](z) - k^2\mathcal{L}[y](z) = \mathcal{L}[f_2(t)](z)$$

$$z^2 \mathcal{L}[y](z) - z\,y(0) - y'(0) - k^2 \mathcal{L}[y](z) = \mathcal{L}[f_2(t)](z)$$

$$z^2 Y(z) - k^2 Y(z) = \frac{4}{z^2(z-2)}$$

$$Y(z)\,(z^2-k^2) = \frac{4}{z^2(z-2)} \Rightarrow Y(z) = \frac{4}{z^2(z-2)} \cdot \frac{1}{z^2-k^2}$$

La solución del problema de valores iniciales viene dado por:

$$y(t) = \mathcal{L}^{-1}[Y(z)](t) = \mathcal{L}^{-1}\left[\frac{4}{z^2(z-2)} \cdot \frac{1}{z^2-k^2}\right](t)$$ es

decir que:

$$\mathcal{L}[y(t)](z) = \frac{4}{z^2(z-2)} \cdot \frac{1}{z^2-k^2}$$

Como: $\mathcal{L}^{-1}\left[\frac{1}{z^2-k^2}\right](t) = \operatorname{Res}\left[\frac{e^{tz}}{z^2-k^2}, k\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2-k^2}, -k\right]$

$$= \frac{e^{kt}}{2k} + \frac{e^{-kt}}{-2k} = \frac{e^{kt} - e^{-kt}}{2k} = \frac{1}{k} \cdot Sh(kt)$$

y $\mathcal{L}^{-1}\left[\frac{4}{z^2(z-2)}\right](t) = \frac{1}{4} - 2t - \frac{1}{4} e^{2t}$

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[\frac{1}{4} - 2t - \frac{1}{4} e^{2t}\right] \cdot \mathcal{L}\left[\frac{1}{k} Sh(kt)\right]$$

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[\left(\frac{1}{4} - 2t - \frac{1}{4} e^{+2t}\right) * \frac{1}{k} Sh(kt)\right](z)$$

La solución del problema de valores iniciales viene dada por:

$y(t) = \left(\frac{1}{4} - 2t - \frac{1}{4} e^{2t}\right) * \frac{1}{k} Sh(kt)$, es decir:

$$y(t) = \frac{1}{k}\int_0^t Sh[k(t-s)] \cdot \left(\frac{1}{4} - 2s - \frac{1}{4} e^{2s}\right) ds$$ (¡menuda integral, fácil pero engorrosa!)

c) $\begin{cases} y'' + 4y' + 13y = f_3(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$f_3(t) = \frac{1}{2}\left(1 - e^{-2t}\right)$

$\mathcal{L}[f_3(t)](z) = F_3(z) = \frac{1}{z(z+2)}$

Solución:

$\mathcal{L}[y''](z) + 4\mathcal{L}[y'](z) + 13\mathcal{L}[y](z) = \mathcal{L}[f_3(t)](z)$

$z^2\mathcal{L}[y](z) - z\,y(0) - y'(0) + 4\left[z\mathcal{L}[y](z) - y(0)\right] + 13\mathcal{L}[y](z) =$

$= \mathcal{L}[f_3(t)](z) \Rightarrow z^2 Y(z) + 4z Y(z) + 13 Y(z) = \frac{1}{z(z+2)}$

$Y(z)\left(z^2 + 4z + 13\right) = \frac{1}{z(z+2)} \Rightarrow Y(z) = \frac{1}{z(z+2)} \cdot \frac{1}{z^2+4z+13}$

Como $Y(z) = \mathcal{L}[y(t)](z)$, siendo $y(t)$ la solución del problema de valores iniciales:

$\mathcal{L}[y(t)](z) = \frac{1}{z(z+2)} \cdot \frac{1}{z^2+4z+13}$

$$\frac{1}{z^2+4z+13} = \frac{1}{[z-(2+3i)][z-(2-3i)]}$$

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+4z+13}\right](t) = \operatorname{Res}\left[\frac{e^{zt}}{z^2+4z+13}, 2+3i\right] + \operatorname{Res}\left[\frac{e^{zt}}{z^2+4z+13}, 2-3i\right]$$

$$p(z) = e^{zt} \quad \begin{cases} p(2+3i) = e^{(2+3i)t} = e^{2t} + e^{3ti} \\ p(2-3i) = e^{(2-3i)t} = e^{2t} \cdot e^{-3ti} \end{cases}$$

$$q(z) = z^2+4z+13 \Rightarrow q'(z) = 2z+4 \Rightarrow \begin{cases} q'(2+3i) = 8+6i \\ q'(2-3i) = 8-6i \end{cases}$$

luego:

$$\mathcal{L}^{-1}\left[\frac{1}{z^2+4z+13}\right](t) = e^{2t}\left[\frac{e^{3ti}}{8+6i} + \frac{e^{-3ti}}{8-6i}\right] =$$

$$= e^{2t}\left[\frac{(8-6i)\,e^{3ti}}{100} + \frac{(8+6i)\,e^{-3ti}}{100}\right] =$$

$$= \frac{e^{2t}}{100} \cdot \left[8\cdot(e^{3ti}+e^{-3ti}) - 6i\cdot(e^{3ti}-e^{-3ti})\right] =$$

$$= \frac{e^{2t}}{100}\left[16 \cdot \frac{e^{3ti}+e^{-3ti}}{2} + 12 \cdot \frac{e^{3ti}-e^{-3ti}}{2i}\right] =$$

$$= \frac{e^{2t}}{100}\left(16 \cos t + 12 \operatorname{sen} t\right)$$

Entonces:

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[\frac{1}{2}(1-e^{-2t})\right](z)\cdot\mathcal{L}\left[\frac{e^{2t}}{100}(16\cos t+12\,\mathrm{sen}\,t)\right](z)$$

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[\frac{1}{2}(1-e^{-2t}) * \frac{e^{2t}}{100}\cdot(16\cos t+12\,\mathrm{sen}\,t)\right](z)$$

La solución del problema de valores iniciales es:

$$y(t) = \frac{1}{2}(1-e^{-2t}) * \frac{e^{2t}}{100}(16\cos t+12\,\mathrm{sen}\,t)$$

$$y(t) = \int_0^t \frac{1}{2}\left(1-e^{-2(t-s)}\right)\cdot\frac{e^{2s}}{100}(16\cos s+12\,\mathrm{sen}\,s)\,ds$$

$$y(t) = \frac{1}{200}\int_0^t \left(e^{2s}-e^{4s-2t}\right)(16\cos s+12\,\mathrm{sen}\,6)\,ds$$

d) $\begin{cases} y''+6y'+9y = f_4(t) \\ y(0) = A \\ y'(0) = B \end{cases}$ $\qquad f_4(t) = -\frac{1}{2}(t\cos t-\mathrm{sen}\,t)$

$$\mathcal{L}[f_4(t)] = F_4(z) = \frac{1}{(z^2+1)^2}$$

Solución:

$$\mathcal{L}[y''](z)+6\,\mathcal{L}[y'](z)+9\,\mathcal{L}[y](z) = \mathcal{L}[f_4(t)](z)$$

$$z^2\mathcal{L}[y(t)]-zy(0)-y'(0)+6\left[z\mathcal{L}[y](z)-y(0)\right]+9\mathcal{L}[y(t)] = \frac{1}{(z^2+1)^2}$$

$$z^2 Y(z) - Az - B + 6\left[zY(z) - A \right] + 9Y(z) = \frac{1}{(z^2+1)^2}$$

$$Y(z)\left[z^2 + 6z + 9 \right] - (Az + 6A + B) = \frac{1}{(z^2+1)^2}$$

$$Y(z)\left[z^2+6z+9 \right] = \frac{1}{(z^2+1)^2} + Az+6A+B$$

$$Y(z) = \frac{1+(Az+6A+B)(z^2+1)^2}{(z^2+1)^2 \cdot (z^2+6z+9)}$$

$$Y(z) = \frac{1}{(z^2+1)^2} \cdot \frac{(Az+6A+B)(z^2+1)^2}{(z^2+6z+9)}$$

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[-\tfrac{1}{2}(t\cos t - \operatorname{sen} t)\right](z) \cdot \mathcal{L}\left[\xi(t)\right]$$

No puedo seguir, ¿como hallamos una funcion $\xi(t)$ tal que $\mathcal{L}[\xi(t)] = \dfrac{(Az+6A+B)(z^2+1)^2}{z^2+6z+9}$? esto es un galimatias ya que desconozco el valor de A y B. Conceptualmente hablando es fácil, lo malo son los calculos. De todas formas voy a intentarlo, puede que tenga interés ya que la fracción que nos surge es racional impropia (grado del numerador mayor que el grado del denominador)

$$(Az+6A+B)(z^4+2z^2+1) = Az^5+6Az^4+Bz^4+2Az^3+12Az^2+2Bz^2+$$
$$+Az+6A+B = Az^5+(6A+B)z^4+2Az^3+(12A+2B)z^2+Az+(6A+B)$$

$$\begin{array}{l|l}
\cancel{Az^5}+(6A+B)z^4+2Az^3+(12A+2B)z^2+Az+6A+B & z^2+6z+9 \\ \cline{2-2}
-\cancel{Az^5}-6Az^4-9Az^3 & Az^3+Bz^2-(7A+6B)z+ \\
\quad \cancel{Bz^4}-7Az^3+(12A+2B)z^2 & +(54A+29B) \\
\quad -\cancel{Bz^4}-6Bz^3-9Bz^2 & \\
\qquad -\cancel{(7A+6B)z^3}+(12A-7B)z^2+Az & \\
\qquad +\cancel{(7A+6B)z^3}+(42A+36B)z^2+(42A+54B)z & \\
\qquad\quad \cancel{(54A+29B)}z^2+(43A+54B)z+6A+B & \\
\qquad\quad -\cancel{(54A+29B)}z^2-(324A+174B)z-486A-261B & \\
\qquad\qquad -\underbrace{(281A+120B)}_{=\alpha}z-\underbrace{(482A+260B)}_{=\beta} &
\end{array}$$

Luego:

$$\frac{(Az+6A+B)(z^2+1)^2}{(z^2+6z+9)} = Az^3+Bz^2-(7A+6B)z+(54A+29B)+\frac{\alpha z+\beta}{z^2+6z+9}$$

Por tanto:

$$\mathcal{L}^{-1}\left[\frac{(Az+6A+B)(z^2+1)^2}{z^2+6z+9}\right](t) = A\cdot\mathcal{L}^{-1}[z^3](t)+B\cdot\mathcal{L}^{-1}[z^2](t) -$$
$$-(7A+6B)\mathcal{L}^{-1}[z](t)+(54A+29B)\mathcal{L}^{-1}[1](t)+\mathcal{L}^{-1}\left[\frac{\alpha z+\beta}{z^2+6z+9}\right](t) =$$
$$A\cdot t^4+Bt^3-(7A+6B)t^2+(54A+29B)t+\mathcal{L}^{-1}\left[\frac{\alpha z+\beta}{z^2+6z+9}\right](t)$$

Ahora calculamos $\mathcal{L}^{-1}\left[\frac{\alpha z+\beta}{z^2+6z+9}\right](t)$

$$\mathcal{L}^{-1}\left[\frac{\alpha z+\beta}{z^2+6z+9}\right](t) = \operatorname{Res}\left[\frac{e^{tz}(\alpha z+\beta)}{(z+3)^2}, -3\right] =$$

$$= \lim_{z\to -3} \frac{d}{dz}\left[\cancel{(z+3)^2}\frac{e^{tz}(\alpha z+\beta)}{\cancel{(z+3)^2}}\right] = \lim_{z\to -3}\left[te^{tz}(\alpha z+\beta)+\alpha e^{tz}\right]$$

$$= t\cdot e^{-3t}(-3\alpha+\beta)+\alpha e^{-3t}$$

Luego:

$$\mathcal{L}^{-1}\left[\frac{(Az+6A+B)(z^2+1)^2}{z^2+6z+9}\right](t) = At^4+Bt^3-(7A+6B)t^2+(54A+29B)t +$$

$$+ te^{-3t}(-3\alpha+\beta)+\alpha\cdot e^{-3t} = \xi(t).$$

Entonces:

$$\mathcal{L}[y(t)](z) = \mathcal{L}\left[-\frac{1}{2}(t\cos t-\operatorname{sen} t)\right](z)\cdot\mathcal{L}[\xi(t)](z)$$

con lo que:

$$y(t) = -\frac{1}{2}(t\cos t-\operatorname{sen} t) * \xi(t)$$

17) Resolver las siguientes ecuaciones:

a) $f_1(t) = 1 + 2 \int_0^t f_1(t-s) \cdot e^{-2s} ds$

Solución:

Podemos poner $f_1(t) = 1 + 2 f_1(t) * e^{-2t}$. Si aplicamos la transformada de Laplace:

$\mathcal{L}[f_1(t)](z) = \mathcal{L}[1](z) + 2 \mathcal{L}[f_1(t) * e^{-2t}]$, es decir

$$\mathcal{L}[f_1(t)](z) = \mathcal{L}[1](z) + 2 \mathcal{L}[f_1(t)](z) \cdot \mathcal{L}[e^{-2t}](z)$$

$$\mathcal{L}[f_1(t)](z) = \frac{1}{z} + 2 \cdot \mathcal{L}[f_1(t)](z) \cdot \frac{1}{z+2}$$

$$\left(1 - \frac{2}{z+2}\right) \mathcal{L}[f_1(t)](z) = \frac{1}{z}$$

$$\frac{z}{z+2} \cdot \mathcal{L}[f_1(t)](z) = \frac{1}{z} \Rightarrow \mathcal{L}[f_1(t)](z) = \frac{z+2}{z^2}$$

Por lo tanto:

$$f_1(t) = \mathcal{L}^{-1}\left[\frac{z+2}{z^2}\right](t) = \operatorname{Res}\left[\frac{(z+2)e^{tz}}{z^2}, 0\right] =$$

$$= \lim_{z \to 0} \frac{d}{dz}\left[(z+2)e^{tz}\right] = \lim_{z \to 0} \left(e^{tz} + t(z+2)e^{tz}\right) = 1 + 2t$$

Luego $f(t) = 1 + 2t$

b) $f_2(t) = 1 + \int_0^t f_2(s) \cdot \operatorname{sen}(t-s)\, ds$

Solución:

$f_2(t) = 1 + f_2(t) * \operatorname{sen} t$. Por tanto:

$$\mathcal{L}[f_2(t)](z) = \mathcal{L}[1](z) + \mathcal{L}[f_2(t) * \operatorname{sen} t](z)$$

$$\mathcal{L}[f_2(t)](z) = \frac{1}{z} + \mathcal{L}[f_2(t)](z) \cdot \mathcal{L}[\operatorname{sen} t](z)$$

$$\mathcal{L}[f_2(t)](z) = \frac{1}{z} + \mathcal{L}[f_2(t)](z) \cdot \frac{1}{z^2+1}$$

$$\left[1 - \frac{1}{z^2+1}\right] \mathcal{L}[f_2(t)](z) = \frac{1}{z}$$

$$\frac{z^2}{z^2+1} \cdot \mathcal{L}[f_2(t)](z) = \frac{1}{z} \Rightarrow \mathcal{L}[f_2(t)] = \frac{z^2+1}{z^3} \Rightarrow$$

$$\Rightarrow f_2(t) = \mathcal{L}^{-1}\left[\frac{z^2+1}{z^3}\right](t) = \operatorname{Res}\left[\frac{(z^2+1)e^{tz}}{z^3}, 0\right] =$$

$$= \lim_{z \to 0} \frac{d}{dz}\left[\cancel{z^3} \cdot \frac{(z^2+1) \cdot e^{tz}}{\cancel{z^3}}\right] = \lim_{z \to 0} \left(2z \cdot e^{tz} + t \cdot (z^2+1) e^{tz}\right)$$

$$= t$$

Luego $f_2(t) = t$.

c) $f_3(t) = t + \int_0^t f_3(t-s)\cdot e^{-s}\,ds$

Podemos poner $f_3(t) = t + f_3(t) * e^{-t}$. Aplicando transformadas de Laplace:

$$\mathcal{L}[f_3(t)](z) = \mathcal{L}[t](z) + \mathcal{L}[f_3(t)](z)\cdot\mathcal{L}[e^{-t}](z)$$

$$\mathcal{L}[f_3(t)](z) = \frac{1}{z^2} + \mathcal{L}[f_3(t)](z)\cdot\frac{1}{z+1}$$

$$\left(1-\frac{1}{z+1}\right)\mathcal{L}[f_3(t)](z) = \frac{1}{z^2}$$

$$\frac{z}{z+1}\cdot\mathcal{L}[f_3(t)](z) = \frac{1}{z^2} \Rightarrow \mathcal{L}[f_3(t)](z) = \frac{z+1}{z^3} \Rightarrow$$

$$\Rightarrow f_3(t) = \mathcal{L}^{-1}\left[\frac{z+1}{z^3}\right] = \operatorname{Res}\left[\frac{e^{tz}\cdot(z+1)}{z^3}, 0\right] =$$

$$= \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3\cdot\frac{e^{tz}(z+1)}{z^3}\right] = \lim_{z\to 0}\frac{1}{2}\cdot t\cdot e^{tz}(tz+t+2)$$

$= \frac{1}{2}(t^2+2t)$. Luego $f_3(t) = \frac{1}{2}(t^2+2t)$

Los apartados d, e, f son similares, me los salto.

Voy a pasar al apartado g)

g) $f_7(t) = t^2 - 2\int_0^t f_7(t-s)\cdot \mathrm{Sh}\, 2s\, ds$

$f_7(t) = t^2 - 2 f_7(t) * \mathrm{Sh}\, t$. Aplicando transformada de Laplace y teniendo en cuenta que $\mathcal{L}[\mathrm{Sh}\, t](z) = \dfrac{1}{z^2-1}$

$$\mathcal{L}[f_7(t)](z) = \mathcal{L}[t^2](z) - 2\mathcal{L}[f_7(t)](z)\cdot \mathcal{L}[\mathrm{Sh}\, t](z)$$

$$\mathcal{L}[f_7(t)](z) = \frac{2!}{z^3} - 2\mathcal{L}[f_7(t)](z) - \frac{1}{z^2-1}$$

$$\mathcal{L}[f_7(t)](z)\left(1 + \frac{2}{z^2-1}\right) = \frac{2}{z^3}$$

$$\mathcal{L}[f_7(t)](z) = \frac{z^2-1}{z^2+1}\cdot\frac{2}{z^3} = \frac{2(z^2-1)}{z^3(z^2+1)}$$

Luego:

$$f_7(t) = \mathcal{L}^{-1}\left[\frac{2(z^2-1)}{z^3(z^2+1)}\right] = 2\cdot\mathcal{L}^{-1}\left[\frac{z^2-1}{z^3(z^2+1)}\right] =$$

$$2\left[\mathrm{Res}\left(\frac{(z^2-1)e^{tz}}{z^3(z^2+1)}, 0\right) + \mathrm{Res}\left[\frac{(z^2-1)\cdot e^{tz}}{z^3(z^2+1)}, i\right] + \mathrm{Res}\left[\frac{(z^2-1)e^{t}}{z^3(z^2+1)}, -i\right]\right]$$

$$\Rightarrow f_7(t) = 2\left[4 - t^2 - (e^{ti} - e^{-ti})\right] \Rightarrow f_7(t) = 2(4 - t^2 - 2i\,\mathrm{sen}\, t)$$

$$\mathrm{Res}\left[\frac{(z^2-1)\cdot e^{tz}}{z^3(z^2+1)}, 0\right] = \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^2-1}{z^2+1}\cdot e^{tz}\right] =$$

$$= \frac{1}{2}\lim_{z\to 0}\left(\frac{-4z^2+4}{(z^2+1)^2}e^{tz} + \frac{4z}{z^2+1}\, t\cdot e^{tz} + \frac{4z}{z^2+1}\cdot e^{tz} + \frac{z^2-1}{z^2+1}\, e^{tz}t^2\right) =$$

Luego:

$$\text{Res}\left[\frac{(z-1)\cdot e^{tz}}{z^3(z^2+1)}, 0\right] = 4 - t^2$$

$$\text{Res}\left[\frac{(z^2-1)\cdot e^{tz}}{z^3(z^2+1)}, i\right] = -e^{ti} \quad ; \quad \text{Res}\left[\frac{(z^2-1)\cdot e^{tz}}{z^3(z^2+1)}, -i\right] = e^{-ti}$$

$$p(z) = (z^2-1)\,e^{tz} \Rightarrow p(i) = -2e^{ti} \quad p(-i) = -2\,e^{-ti}$$

$$q(z) = z^5 + z^3 \Rightarrow q'(z) = 5z^4 + 3z^2 \Rightarrow q'(i) = 2 \quad q'(-i) = 2$$

El apartado i) me lo salto, es igual que el h) que ahora hacemos:

h) $f_8(t) = 1 + 2\displaystyle\int_0^t f_8(t-s)\cos s\,ds$

Podemos poner:

$f_8(t) = 1 + 2\,f_8(t) * \cos t$. Teniendo en cuenta que

$\mathcal{L}[\cos t](z) = \dfrac{z}{z^2+1}$ $\text{Re}\, z > 0$, aplicando transformadas de Laplace:

$$\mathcal{L}[f_8(t)](z) = \mathcal{L}[1](z) + 2\,\mathcal{L}[f_8(t)]\cdot\mathcal{L}[\cos t](z)$$

$$\mathcal{L}[f_8(t)](z) = \frac{1}{z} + \frac{2z}{z^2+1}\,\mathcal{L}[f_8(t)](z)$$

$$\left(1 - \frac{2z}{z^2+1}\right) \mathcal{L}[f_8(t)](z) = \frac{1}{z}$$

$$\frac{z^2-2z+1}{z^2+1} \cdot \mathcal{L}[f_8(t)](z) = \frac{1}{z}$$

$$\frac{(z-1)^2}{z^2+1} \cdot \mathcal{L}[f_8(t)](z) = \frac{1}{z}$$

$$\mathcal{L}[f_8(t)](z) = \frac{z^2+1}{z(z-1)^2} \Rightarrow f_8(t) = \mathcal{L}^{-1}\left[\frac{z^2+1}{z(z-1)^2}\right](t) \Rightarrow$$

$$\Rightarrow f_8(t) = \operatorname{Res}\left[\frac{(z^2+1)\,e^{tz}}{z(z-1)^2}, 0\right] + \operatorname{Res}\left[\frac{(z^2+1)\,e^{tz}}{z(z-1)^2}, 1\right] =$$

$$\Rightarrow f_8(t) = -1 + 2e^t$$

18) Resolver los siguientes sistemas de ecuaciones utilizando la transformada de Laplace:

a) $\left.\begin{aligned} &2x' + 2x + y' - y = 3t \\ &x' + x + y' + y = 1 \\ &x(0) = 1 \\ &y(0) = 3 \end{aligned}\right\}$

Si restamos a la primera ecuación la segunda:

$$x' + x - 2y = 3t - 1 \Rightarrow x' = -x + 2y + 3t - 1$$

Si sustituimos x' en la segunda ecuación:

$$\cancel{-x} + 2y + 3t - 1 + \cancel{x} + y' + y = 1 \;;\; y' = -3y - 3t + 2$$

El sistema queda en esta otra forma equivalente:

$$\left.\begin{aligned} x' &= -x + 2y + 3t - 1 \\ y' &= \quad\; -3y - 3t + 2 \end{aligned}\right\}$$

En forma matricial se expresa como:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3t - 1 \\ -3t + 2 \end{pmatrix}$$

Llamando $y = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$:

$y' = Ay + B$ donde $A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 3t-1 \\ -3t+2 \end{pmatrix}$

Aplicando transformada de Laplace:

$$\mathcal{L}[y'](z) = A \cdot \mathcal{L}[y](z) + \mathcal{L}[B](z)$$

$$z \cdot \mathcal{L}[y(z)] - y(0) = A \cdot \mathcal{L}[y](z) + \mathcal{L}[B](z)$$

Llamando $Y(z) = \mathcal{L}[y](z)$

$$z \cdot Y(z) - y(0) = A \cdot Y(z) + \mathcal{L}[B](z)$$

$$(zI - A)\, Y(z) = y(0) + \mathcal{L}[B](z)$$

$$\mathcal{L}[B](z) = \begin{pmatrix} 3\mathcal{L}[t](z) - \mathcal{L}[1](z) \\ -3\mathcal{L}[t](z) + 2\mathcal{L}[1](z) \end{pmatrix} = \begin{pmatrix} \frac{3}{z^2} - \frac{1}{z} \\ \frac{-3}{z^2} + \frac{2}{z} \end{pmatrix}$$

Por tanto:

$$(zI - A)\, Y(z) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{3}{z^2} - \frac{1}{z} \\ \frac{-3}{z^2} + \frac{2}{z} \end{pmatrix} = \begin{pmatrix} \frac{z^2 - z + 3}{z^2} \\ \frac{3z^2 + 2z - 3}{z^2} \end{pmatrix}$$

con lo que:

$$Y(z) = (zI - A)^{-1} \cdot \frac{1}{z^2} \begin{pmatrix} z^2 - z + 3 \\ 3z^2 + 2z - 3 \end{pmatrix}$$

Calculemos $(zI - A)^{-1} = \begin{pmatrix} z+1 & -2 \\ 0 & z-3 \end{pmatrix}^{-1}$.

$$(zI-A)^t = \begin{pmatrix} z+1 & 0 \\ -2 & z-3 \end{pmatrix} ; \quad |zI-A| = z^2-2z-3$$

$$\text{Adj}\,(zI-A)^t = \begin{pmatrix} z-3 & 2 \\ 0 & z+1 \end{pmatrix} \Rightarrow (zI-A)^{-1} = \frac{1}{z^2-2z-3} \begin{pmatrix} z-3 & 2 \\ 0 & z+1 \end{pmatrix}$$

Luego:

$$Y(z) = \frac{1}{z^2-2z-3} \begin{pmatrix} z-3 & -2 \\ 0 & z+1 \end{pmatrix} \cdot \frac{1}{z^2} \begin{pmatrix} z^2-z+3 \\ 3z^2+2z-3 \end{pmatrix}$$

$$Y(z) = \frac{1}{z^2(z^2-2z-3)} \begin{pmatrix} z-3 & -2 \\ 0 & z+1 \end{pmatrix} \begin{pmatrix} z^2-z+3 \\ 3z^2+2z-3 \end{pmatrix} =$$

$$Y(z) = \frac{1}{z^2(z^2-2z-3)} \begin{pmatrix} z^3-10z^2+2z-3 \\ 3z^3+5z^2-z-3 \end{pmatrix}$$

$$Y(z) = \begin{pmatrix} \dfrac{z^3-10z^2+2z-3}{z^2(z^2-2z-3)} \\[2ex] \dfrac{3z^3+5z^2-z-3}{z^2\cdot(z^2-2z-3)} \end{pmatrix} \Rightarrow y(t) = \mathcal{L}^{-1}\left[\begin{pmatrix} \dfrac{z^3-10z^2+2z-3}{z^2(z^2-2z-3)} \\[2ex] \dfrac{3z^3+5z^2-z-3}{z^2(z^2-2z-3)} \end{pmatrix} \right](t)$$

$$\mathcal{L}^{-1}\left[\frac{z^3-10z^2+2z-3}{z^2(z^2-2z-3)} \right] = \text{Res}\left[\frac{e^{tz}(z^3-10z^2+2z-3)}{z^2(z^2-2z-3)}, 0 \right] +$$

$$+ \text{Res}\left[\frac{e^{tz}(z^3-10z^2+2z-3)}{z^2(z^2-2z-3)}, 3 \right] + \text{Res}\left[\frac{e^{tz}(z^3-10z^2+2z-3)}{z^2(z^2-2z-3)}, -1 \right]$$

Efectuando los calculos obtenemos que:

$$\mathcal{L}^{-1}\left[\frac{z^3-10z^2+2z-3}{z^2(z^2-2z-3)}\right](t)=\frac{t}{3}+\frac{5e^{3t}}{3}+4\cdot e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{3z^3+5z^2-z-3}{z^2(z^2-2z-3)}\right](t)=\operatorname{Res}\left[\frac{e^{tz}(3z^3+5z^2-z-3)}{z^2(z^2-2z-3)},0\right]+$$

$$+\operatorname{Res}\left[\frac{e^{tz}(3z^3+5z^2-z-3)}{z^2(z^2-2z-3)},-1\right]+\operatorname{Res}\left[\frac{e^{tz}(3z^3+5z^2-z-3)}{z^2(z^2-2z-3)},3\right]=$$

$$=\frac{9t-3}{9}+e^{-t}+\frac{63}{18}e^{3t}$$

Si no me he equivocado la solución del sistema

es:

$$\vec{y}(t)=\begin{pmatrix}\frac{t}{3}+\frac{5e^{3t}}{3}+4e^{-t}\\ \\ \frac{9t-3}{9}+e^{-t}+\frac{63}{18}e^{3t}\end{pmatrix},\quad \text{es decir:}$$

$$x(t)=\frac{t}{3}+\frac{5}{3}e^{3t}+4e^{-t}$$

$$y(t)=t-\frac{1}{3}+e^{-t}+\frac{63}{18}e^{3t}$$

El ejercicio 18 b) es análogo, no lo salto. Los ejercicios de circuitos HAZLOS TU (los tienes resueltos en los apuntes)